

# A THIRD-ORDER DISPERSIVE FLOW FOR CLOSED CURVES INTO KÄHLER MANIFOLDS

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**ABSTRACT.** This paper is devoted to studying the initial value problem for a third-order dispersive equation for closed curves into Kähler manifolds. This equation is a geometric generalization of a two-sphere valued system modeling the motion of vortex filament. We prove the local existence theorem by using geometric analysis and classical energy method.

## 1. INTRODUCTION

In this paper we study the initial value problem of a third-order dispersive flow describing the motion of closed curves on Kähler manifolds. Let  $(N, J, g)$  be a Kähler manifold with an almost complex structure  $J$  and a Kähler metric  $g$ , and let  $\nabla$  be the Levi-Civita connection with respect to  $g$ . Consider the initial value problem of the form

$$u_t = a \nabla_x^2 u_x + J_u \nabla_x u_x + b g_u(u_x, u_x) u_x \quad \text{in } \mathbb{R} \times \mathbb{T}, \quad (1.1)$$

$$u(0, x) = u_0(x) \quad \text{in } \mathbb{T}, \quad (1.2)$$

where  $a, b \in \mathbb{R}$  are constants,  $u = u(t, x)$  is an  $N$ -valued unknown function of  $(t, x) \in \mathbb{R} \times \mathbb{T}$ ,  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ ,  $u_t(t, x) = du_{(t,x)}(\partial/\partial t)_{(t,x)}$ ,  $u_x(t, x) = du_{(t,x)}(\partial/\partial x)_{(t,x)}$ ,  $du$  is the differential of the mapping  $u$ ,  $\nabla_x$  is the covariant derivative on  $u^{-1}TN$  induced from  $\nabla$  with respect to  $x$ , and  $J_u$  and  $g_u$  mean the almost complex structure and the metric at  $u \in N$  respectively. Here  $u^{-1}TN = \bigcup_{(t,x) \in \mathbb{R} \times \mathbb{T}} T_{u(t,x)}N$  is the pull-back bundle over  $\mathbb{R} \times \mathbb{T}$  from  $TN$  via the mapping  $u$ .  $V$  is said to be a section of  $u^{-1}TN$  over  $\mathbb{T} \times \mathbb{R}$  if  $V(t, x) \in T_{u(t,x)}N$  for all  $(t, x) \in \mathbb{R} \times \mathbb{T}$ .  $J_u$  and  $g_u$  are a  $(1,1)$ -tensor field and a  $(0,2)$ -tensor field along  $u$  respectively, and the equation (1.1) is an equality of sections of  $u^{-1}TN$ . We call the solution of (1.1) a dispersive flow in this paper. In particular, when  $a = b = 0$ , the solutions are called one-dimensional Schrödinger maps.

Examples of (1.1) arise in classical mechanics related to vortex filament, ferromagnetic spin chain system and etc. For  $\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$  and  $\vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ , let

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3, \quad |\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}},$$

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

Let  $\mathbb{S}^2$  be the two-sphere in  $\mathbb{R}^3$ , that is,  $\mathbb{S}^2 = \{\vec{u} \in \mathbb{R}^3 \mid |\vec{u}| = 1\}$ . In 1906, Da Rios formulated the equation modeling the motion of vortex filament of the form

$$\vec{u}_t = \vec{u} \times \vec{u}_{xx}, \quad (1.3)$$

where  $\vec{u}(t, x)$  is  $\mathbb{S}^2$ -valued. See his celebrated paper [2], and other references [7] and [10] for instance. The physical model (1.3) is an example of the equation of the one-dimensional

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Schrödinger map. Our equation (1.1) in the setting  $b = a/2$  geometrically generalizes an  $\mathbb{S}^2$ -valued physical model

$$\vec{u}_t = \vec{u} \times \vec{u}_{xx} + a \left[ \vec{u}_{xxx} + \frac{3}{2} \{ \vec{u}_x \times (\vec{u} \times \vec{u}_x) \}_x \right] \quad (1.4)$$

proposed by Fukumoto and Miyazaki in [5].

Here we state the known results on the mathematical analysis of the IVP (1.1)-(1.2). In case  $a = b = 0$ , there are many studies on the existence theorem for (1.1)-(1.2). See [1], [3] [8], [11], [12], [13], [16], [18] and references therein. In [18] Sulem, Sulem and Bardos treated the higher dimensional ferromagnetic spin system of the form

$$\vec{u}_t = \vec{u} \times \Delta_{\mathbb{R}^m} \vec{u}, \quad (1.5)$$

where  $\vec{u}(t, x)$  is the  $\mathbb{S}^2$ -valued function of  $(t, x) \in \mathbb{R} \times \mathbb{R}^m$ ,  $\Delta_{\mathbb{R}^m}$  is the Euclid Laplacian on  $\mathbb{R}^m$ . They proved global existence of smooth solution with small initial data, whereas they also proved that the problem admits the time-global solution with large data only if  $m = 1$ . In [8] Koiso proved the local existence theorem of the IVP for the one-dimensional Schrödinger map for closed curves into Kähler manifolds of the form

$$u_t = J_u \nabla_x u_x \quad (1.6)$$

in  $H^{m+1}(\mathbb{T}; N)$  for any integer  $m \geq 2$ . Furthermore, he proved that the problem admits time-global solution if  $N$  is a locally hermitian symmetric space. Recently, higher dimensional Schrödinger map into Kähler manifolds has been studied. This equation is not only the higher dimensional version of (1.6), but also the geometric generalization of (1.5). See e.g. [3], [11] and [16] for the detail.

In case  $a \neq 0$ , only  $\mathbb{S}^2$ -valued dispersive flow has been studied. In [15] Nishiyama and Tani proved the global existence theorem of the IVP for (1.4) in  $H^{m+1}(\mathbb{T}; \mathbb{R}^3)$  with an integer  $m \geq 2$  in the setting  $b = a/2$ . Moreover, they formulated the IVP for curves with two fixed edges on  $\mathbb{S}^2$  at  $x = \pm\infty$  for  $x \in \mathbb{R}$ , and proved the global existence results also.

The purpose of this paper is to study the existence theorem of (1.1)-(1.2) especially in the setting that  $a \neq 0$ ,  $b \in \mathbb{R}$  and  $N$  is a general Kähler manifold. To state our result, we now introduce some definitions related to Sobolev spaces for mappings.

**Definition 1.1.** Let  $(N, g)$  be a Riemannian manifold, and let  $\mathbb{N}$  be the set of positive integers. For  $m \in \mathbb{N} \cup \{0\}$ , a bundle-valued Sobolev space of mappings is defined by

$$H^{m+1}(\mathbb{T}; N) = \{u \mid u(x) \in N \text{ a.e. } x \in \mathbb{T}, \text{ and } u_x \in H^m(\mathbb{T}; TN)\},$$

where  $u_x \in H^m(\mathbb{T}; TN)$  means that  $u_x$  satisfies

$$\|u_x\|_{H^m(\mathbb{T}; TN)}^2 = \sum_{j=0}^m \int_0^1 g_{u(x)}(\nabla_x^j u_x(x), \nabla_x^j u_x(x)) dx < +\infty.$$

Moreover, let  $I$  be an interval in  $\mathbb{R}$ , and let  $w : (N, g) \rightarrow (\mathbb{R}^d, g_0)$  be an isometric embedding. Here  $g_0$  is the standard Euclidean metric on  $\mathbb{R}^d$ . We say that  $u \in C(I; H^{m+1}(\mathbb{T}; N))$  if  $u(t) \in H^{m+1}(\mathbb{T}; N)$  for all  $t \in I$  and  $w \circ u \in C(I; H^{m+1}(\mathbb{T}; \mathbb{R}^d))$ , where  $C(I; H^{m+1}(\mathbb{T}; \mathbb{R}^d))$  is the set of usual Sobolev space valued continuous functions on  $I$ .

Our main results are the following.

**Theorem 1.1.** *Let  $m \geq 2$  be an integer. Then for any  $u_0 \in H^{m+1}(\mathbb{T}; N)$ , there exists a constant  $T > 0$  depending only on  $a, b, N$  and  $\|u_0\|_{H^2(\mathbb{T}; TN)}$  such that the initial value problem (1.1)-(1.2) possesses a unique solution  $u \in C([-T, T]; H^{m+1}(\mathbb{T}; N))$ .*

Roughly speaking, Theorem 1.1 says that (1.1)-(1.2) has a time-local solution in the usual Sobolev space  $H^3$ . In addition,  $m = 2$  is the smallest integer for (1.1) to make sense in the class  $C([-T, T]; L^2(\mathbb{T}; TN))$ .

We cannot prove any global existence results for (1.1)-(1.2) independently of  $a, b$  and  $N$ . In case  $N = \mathbb{S}^2$ ,  $a \neq 0$  and  $b = a/2$ , Nishiyama and Tani made use of some conservation laws to prove the global existence theorem in [15] and [19]. These conservation laws were discovered by Zakharov and Shabat in the study of the Hirota equation. See [20] for the detail. If we take into account of the effect of the curvature of  $N$  to the third term of (1.1), we obtain the global existence theorem in the same way as [15] and [19] in case that the Kähler manifold  $N$  is a compact Riemann surface with constant Gaussian curvature as a  $C^\infty$ -manifold. We shall prove the following.

**Theorem 1.2.** *Let  $(N, J, g)$  be a compact Riemann surface with constant Gaussian curvature  $K$  and let  $a \neq 0$  and  $b = aK/2$ . Then for any  $u_0 \in H^{m+1}(\mathbb{T}; N)$  with an integer  $m \geq 2$ , there exists a unique solution  $u \in C(\mathbb{R}; H^{m+1}(\mathbb{T}; N))$  to the initial value problem (1.1)-(1.2).*

We remark that Theorem 1.2 generalizes the results of Nishiyama and Tani in [15] and [19]. In other words, the proof of Theorem 1.2 will explain the reason why the global existence theorem of (1.1)-(1.2) holds in case that  $N = \mathbb{S}^2$ . We would also like to recall that there are some classical examples of the compact Riemann surface with constant Gaussian curvature. Indeed, not only two-sphere  $\mathbb{S}^2$  and flat-torus  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ , but also closed orientable surfaces  $\Sigma_g$  with genus  $g \geq 2$  admit the structure of such manifold. The Gaussian curvature  $K$  of them are 1, 0, and  $-1$  respectively.

Our method of the proof of Theorem 1.1 is based on the geometric analysis and the classical energy method. We first remark that the local smoothing effect of dispersive equations breaks down because of the compactness of the domain  $\mathbb{T}$ . See [4] for the detail. Fortunately, however, (1.1) behaves like symmetric hyperbolic systems in some sense, and a geometric classical energy method works for (1.1). More precisely, the Kähler condition  $\nabla J \equiv 0$  and the properties of the Riemannian curvature tensor ensures that the loss of derivatives does not occur in geometric energy estimates. In other words, the solvable structure on the system of partial differential operators comes from the good geometric structures on  $N$ . In addition, we sometimes identify the unknown map  $u$  with  $w \circ u$  via the Nash isometric embedding  $w : (N, J, g) \rightarrow (\mathbb{R}^d, g_0)$  in our proof. It is more convenient to treat the system for  $w \circ u$  than to treat (1.1) directly when we apply the standard argument of partial differential equations.

More concretely, the process of our proof of Theorem 1.1 is as follows. We may assume that  $N$  is compact since the initial curve  $u_0$  lies on a compact subset in  $N$  even if  $N$  is noncompact. It suffices to solve the problem in the positive direction in time. First, we construct a sequence of approximate solutions  $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$  to

$$u_t = -\varepsilon \nabla_x^3 u_x + a \nabla_x^2 u_x + J_u \nabla_x u_x + b g_u(u_x, u_x) u_x \quad \text{in } (0, T_\varepsilon) \times \mathbb{T}, \quad (1.7)$$

$$u(0, x) = u_0(x) \quad \text{in } \mathbb{T}. \quad (1.8)$$

By using a geometric orthogonal decomposition in the tubular neighbourhood of  $w(N)$ , we can check that a kind of maximum principle holds and  $u^\varepsilon(t)$  is  $N$ -valued. Secondly, the geometric classical energy estimates obtain the uniform estimate on the norm and the existence-time of  $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$ . Then the standard compactness argument implies the local existence of solution

$$u \in C([0, T] \times \mathbb{T}; N), \quad u_x \in C([0, T]; H^{m-1}(\mathbb{T}; TN)) \cap L^\infty(0, T; H^m(\mathbb{T}; TN))$$

of (1.1)-(1.2), where  $L^\infty$  is the usual Lebesgue space. Thirdly, we prove the uniqueness of solution by the energy estimate in  $H^1$  of the difference of two solutions with same initial data.

We can choose a good moving frame of the normal bundle of  $w(N)$  in  $\mathbb{R}^d$ , and thus the classical energy method works for the difference of two solutions also. Finally, the continuity in time of  $\nabla_x^m u_x$  in  $L^2(\mathbb{T}; TN)$  can be recovered by the standard argument.

The organization of this paper is as follows. Section 2 is devoted to geometric preliminaries. In Section 3 we construct a sequence of approximate solutions by solving (1.7)-(1.8). In Section 4 we obtain uniform estimates of approximate solutions. In Section 5 we complete the proof of Theorem 1.1. Finally, in Section 6 we prove Theorem 1.2.

## 2. GEOMETRIC PRELIMINARIES

In this section, we introduce notation, recall the relationship between the bundle-valued Sobolev spaces and the standard Sobolev spaces, and obtain the formulation of a system equivalent to (1.1) used later in our proof.

We will use  $C = C(\cdot, \dots, \cdot)$  to denote a positive constant depending on the certain parameters, geometric properties of  $N$ , et al. The partial differentiation is written by  $\partial$ , or the subscript, e.g.,  $\partial_x f$ ,  $f_x$ , to distinguish from the covariant derivative along the curve, e.g.,  $\nabla_x$ . Throughout this paper,  $w$  is an isometric embedding mapping from  $(N, J, g)$  into the standard Euclidean space  $(\mathbb{R}^d, g_0)$ . The existence of  $w$  is ensured by the celebrated works of Nash [14], Gromov and Rohlin [6], and related papers.

Let  $u : \mathbb{T} \rightarrow N$  be given. We denote  $\Gamma(u^{-1}TN)$  by the space of sections of  $u^{-1}TN$  over  $\mathbb{T}$ . For  $V, W \in \Gamma(u^{-1}TN)$ , define  $L^2$ -inner product of them by

$$\int_{\mathbb{T}} g(V, W) dx = \int_0^1 g_{u(x)}(V(x), W(x)) dx,$$

and use the notation

$$\|V\|_{L^2(\mathbb{T}; TN)}^2 = \int_{\mathbb{T}} g(V, V) dx.$$

Then the quantity  $\|u_x\|_{H^m(\mathbb{T}; TN)}^2$  defined in Definition 1.1 is written as

$$\|u_x\|_{H^m(\mathbb{T}; TN)}^2 = \sum_{j=0}^m \|\nabla_x^j u_x\|_{L^2(\mathbb{T}; TN)}^2.$$

At this time we see that  $\|u_x\|_{H^m(\mathbb{T}; TN)} < \infty$  if and only if  $\|(w \circ u)_x\|_{H^m(\mathbb{T}; \mathbb{R}^d)} < \infty$ . See, e.g., [17, Section 1] or [11, Proposition 2.5] on this equivalence. Noting this equivalence and the compactness of  $\mathbb{T}$ , we have

$$\begin{aligned} H^{m+1}(\mathbb{T}; N) &= \{ u \mid u(x) \in N \text{ a.e. } x \in \mathbb{T}, \text{ and } (w \circ u)_x \in H^m(\mathbb{T}; \mathbb{R}^d) \} \\ &= \{ u \mid u(x) \in N \text{ a.e. } x \in \mathbb{T}, \text{ and } w \circ u \in H^{m+1}(\mathbb{T}; \mathbb{R}^d) \}. \end{aligned}$$

We will make use of fundamental Sobolev space theory of  $H^{m+1}(\mathbb{T}; \mathbb{R}^d)$  later in our proof.

Set  $I = [-T, T]$  for  $T > 0$ . The equation (1.1) is equivalent to a system for  $w \circ u$  as follows.

**Lemma 2.1.** *Assume that  $m \geq 2$  is an integer. Then  $u \in C(I; H^{m+1}(\mathbb{T}; N))$  satisfies (1.1)-(1.2) if and only if  $v = w \circ u \in C(I; H^{m+1}(\mathbb{T}; \mathbb{R}^d))$  is  $w(N)$ -valued and satisfies*

$$\begin{aligned} v_t &= a\{v_{xxx} + [A(v)(v_x, v_x)]_x + A(v)(v_{xx} + A(v)(v_x, v_x), v_x)\} \\ &\quad + \tilde{J}_v(v_{xx} + A(v)(v_x, v_x)) + b|v_x|^2 v_x \end{aligned} \quad \text{in } I \times \mathbb{T}, \quad (2.1)$$

$$v(0, x) = w \circ u_0(x) \quad \text{in } \mathbb{T}. \quad (2.2)$$

Here,  $A(v)(\cdot, \cdot) : T_v w(N) \times T_v w(N) \rightarrow (T_v w(N))^\perp$  is the second fundamental form of  $w(N) \subset \mathbb{R}^d$  and  $\tilde{J}_v = dw_{w^{-1} \circ v} J_{w^{-1} \circ v} dw_v^{-1}$  on  $T_v w(N)$  at  $v \in w(N)$  respectively.

*Proof of Lemma 2.1.* Suppose  $u \in C(I; H^{m+1}(\mathbb{T}; N))$  satisfies (1.1)-(1.2). Since  $m \geq 2$ , the mapping  $v = w \circ u : I \times \mathbb{T} \rightarrow w(N)$  satisfies

$$v_t = (w \circ u)_t = dw_u(u_t) = a dw_u(\nabla_x^2 u_x) + dw_u(J_u \nabla_x u_x) + b dw_u(g_u(u_x, u_x)u_x)$$

in  $C(I; L^2(\mathbb{T}; \mathbb{R}^d))$ . Moreover we deduce

$$dw_u(\nabla_x u_x) = v_{xx} + A(v)(v_x, v_x), \quad (2.3)$$

$$\begin{aligned} dw_u(\nabla_x^2 u_x) &= [dw_u(\nabla_x u_x)]_x + A(v)(dw_u(\nabla_x u_x), v_x) \\ &= v_{xxx} + [A(v)(v_x, v_x)]_x + A(v)(v_{xx} + A(v)(v_x, v_x), v_x), \end{aligned} \quad (2.4)$$

$$dw_u(g_u(u_x, u_x)u_x) = g(u_x, u_x)dw_u(u_x) = |v_x|^2 v_x \quad (2.5)$$

from the definition of the covariant derivative on  $u^{-1}TN$  and the isometricity of  $w$ . Combining (2.3), (2.4) and (2.5), we obtain that  $v$  solves (2.1)-(2.2) in the class  $C(I; L^2(\mathbb{T}; \mathbb{R}^d))$ .

Conversely, suppose  $v \in C(I; H^{m+1}(\mathbb{T}; \mathbb{R}^d))$  takes value in  $w(N)$  and solves (2.1)-(2.2). Since  $dw$  is injective, it immediately follows from the same calculus as above that  $u = w^{-1} \circ v \in C(I; H^{m+1}(\mathbb{T}; N))$  solves (2.1)-(2.2) in  $C(I; L^2(\mathbb{T}; TN))$ .  $\square$

### 3. PARABOLIC REGULARIZATION

Assume that  $N$  is compact in this section. The aim of this section is to obtain a sequence  $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$  solving

$$u_t = -\varepsilon \nabla_x^3 u_x + a \nabla_x^2 u_x + J_u \nabla_x u_x + b g_u(u_x, u_x)u_x \quad \text{in } (0, T_\varepsilon) \times \mathbb{T}, \quad (3.1)$$

$$u(0, x) = u_0(x) \quad \text{in } \mathbb{T} \quad (3.2)$$

for each  $\varepsilon \in (0, 1)$ , where  $u = u^\varepsilon(t, x)$  is also an  $N$ -valued unknown function of  $(t, x) \in [0, T_\varepsilon] \times \mathbb{T}$ , and  $u_0$  is the same initial data as that of (1.1)-(1.2) independent of  $\varepsilon \in (0, 1)$ .

In the same way as Lemma 2.1, (3.1)-(3.2) is equivalent to the following problem

$$v_t = -\varepsilon v_{xxxx} + F(v) \quad \text{in } (0, T_\varepsilon) \times \mathbb{T}, \quad (3.3)$$

$$v(0, x) = w \circ u_0(x) \quad \text{in } \mathbb{T}, \quad (3.4)$$

where  $v = v^\varepsilon(t, x)$  is a  $w(N)$ -valued unknown function of  $(t, x) \in [0, T_\varepsilon] \times \mathbb{T}$ . Here

$$\begin{aligned} F(v) &= -\varepsilon \{ [A(v)(v_x, v_x)]_{xx} + [A(v)(v_{xx} + A(v)(v_x, v_x), v_x)]_x \\ &\quad + A(v)(v_{xxx} + [A(v)(v_x, v_x)]_x + A(v)(v_{xx} + A(v)(v_x, v_x), v_x), v_x) \} \\ &\quad + a \{ v_{xxx} + [A(v)(v_x, v_x)]_x + A(v)(v_{xx} + A(v)(v_x, v_x), v_x) \} \\ &\quad + \tilde{J}_v(v_{xx} + A(v)(v_x, v_x)) + b |v_x|^2 v_x. \end{aligned}$$

For  $F(v)$ , notice that there exists  $G \in C^\infty(\mathbb{R}^{4d}; \mathbb{R}^d)$  such that

$$F(v) = G(v, v_x, v_{xx}, v_{xxx}), \quad G(v, 0, 0, 0) = 0,$$

for  $v : \mathbb{T} \rightarrow w(N)$ . Note that (3.3) is a system of fourth-order parabolic equations for  $w(N)$ -valued function and represents the equality of sections of  $v^{-1}Tw(N)$ . We show the following.

**Proposition 3.1.** *Let  $u_0 \in H^{m+1}(\mathbb{T}; N)$  with an integer  $m \geq 2$ . Then for each  $\varepsilon \in (0, 1)$ , there exists a constant  $T_\varepsilon = T(\varepsilon, a, b, N, \|(w \circ u_0)_x\|_{H^m(\mathbb{T}; \mathbb{R}^d)}) > 0$  such that (3.3)-(3.4) has a unique solution  $v = v^\varepsilon \in C([0, T_\varepsilon]; H^{m+1}(\mathbb{T}; \mathbb{R}^d))$  satisfying  $v(t, x) \in w(N)$  for all  $(t, x) \in [0, T_\varepsilon] \times \mathbb{T}$ .*

For the solution  $v$  in Proposition 3.1, the equivalence between (3.1)-(3.2) and (3.3)-(3.4) implies that  $u = w^{-1} \circ v$  solves (3.1)-(3.2). The proof of this proposition consists of the following two steps. We first construct the solution of (3.3)-(3.4) whose image are contained in a tubular neighbourhood of  $w(N)$  in  $\mathbb{R}^d$ . Namely, for  $\delta > 0$ , let  $(w(N))_\delta$  be a  $\delta$ -tubular neighbourhood of  $w(N) \subset \mathbb{R}^d$  defined by

$$(w(N))_\delta = \{Q = (q, X) \in \mathbb{R}^d \mid q \in w(N), X \in (T_q w(N))^\perp, |X| < \delta\},$$

and let  $\pi : (w(N))_\delta \rightarrow w(N)$  be the nearest point projection map defined by  $\pi(Q) = q$  for  $Q = (q, X) \in (w(N))_\delta$ . Since  $w(N)$  is compact, it is well-known that, for any sufficiently small  $\delta$ ,  $\pi$  exists and is smooth. We fix such small  $\delta$ , and construct a unique time-local solution of (3.3)-(3.4) in the class

$$Y_T^{m,\delta} = \{v \in C([0, T]; H^{m+1}(\mathbb{T}; \mathbb{R}^d)) \mid \|v - w \circ u_0\|_{L^\infty((0, T) \times \mathbb{T}; \mathbb{R}^d)} \leq \delta/2\}$$

for sufficiently small  $T > 0$ . The second step is to check that this solution is actually  $w(N)$ -valued by using a kind of maximum principle. In short, it suffices to show the following two lemmas.

**Lemma 3.2.** *For each  $\varepsilon \in (0, 1)$ , there exists a constant  $T_\varepsilon > 0$  depending on  $\varepsilon, a, b, N$  and  $\|(w \circ u_0)_x\|_{H^m(\mathbb{T}; \mathbb{R}^d)}$  and there exists a unique solution  $v \in Y_{T_\varepsilon}^{m,\delta}$  to*

$$v_t = -\varepsilon v_{xxxx} + F(\pi \circ v) \quad \text{in} \quad (0, T_\varepsilon) \times \mathbb{T}, \quad (3.5)$$

$$v(0, x) = w \circ u_0(x) \quad \text{in} \quad \mathbb{T}. \quad (3.6)$$

**Lemma 3.3.** *Fix  $\varepsilon \in (0, 1)$ . Assume that  $v = v^\varepsilon \in Y_{T_\varepsilon}^{m,\delta}$  solves (3.5)-(3.6). Then  $v(t, x) \in w(N)$  for all  $(t, x) \in [0, T_\varepsilon] \times \mathbb{T}$ , thus  $v$  solves (3.3)-(3.4).*

*Proof of Lemma 3.2.* The idea of the proof is due to the contraction mapping argument.

Let  $L$  be a nonlinear map defined by

$$Lv(t) = S_\varepsilon(t)v_0 + \int_0^t S_\varepsilon(t-s)F(\pi \circ v)(s)ds,$$

where  $v_0 = w \circ u_0$  and for  $\psi \in H^{m+1}(\mathbb{T}; \mathbb{R}^d)$

$$S_\varepsilon(t)\psi(x) = \sum_{n=-\infty}^{\infty} e^{-\varepsilon t(2\pi n)^4 + 2\pi i n x} \int_0^1 e^{-2\pi i n y} \psi(y) dy$$

is the solution of the linear problem associated to (3.5)-(3.6). Set  $M = \|v_{0x}\|_{H^m(\mathbb{T}; \mathbb{R}^d)}$ , and define the space

$$Z_T^{m,\delta} = \{v \in Y_T^{m,\delta} \mid \|v_x\|_{L^\infty(0, T; H^m(\mathbb{T}; \mathbb{R}^d))} \leq 2M\},$$

which is a closed subset of the Banach space  $C([0, T]; H^{m+1}(\mathbb{T}; \mathbb{R}^d))$ . To complete the proof, we have only to show that the map  $L$  has a unique fixed point in  $Z_{T_\varepsilon}^{m,\delta}$  for sufficiently small  $T_\varepsilon$ , since the uniqueness in the whole space  $Y_{T_\varepsilon}^{m,\delta}$  follows by similar and standard argument.

The operator  $-\varepsilon \partial_x^4$  gains the regularity of order 3, since  $\varepsilon^{j/4} t^{j/4} |n|^j e^{-\varepsilon t(2\pi n)^4}$  is bounded for  $j = 0, 1, 2, 3$ . In fact, there exists  $C_1 > 0$  such that for any  $\psi \in H^{m-2}(\mathbb{T}; \mathbb{R}^d)$

$$\|S_\varepsilon(t)\psi\|_{H^{m+1}(\mathbb{T}; \mathbb{R}^d)} \leq C_1 \varepsilon^{-3/4} t^{-3/4} \|\psi\|_{H^{m-2}(\mathbb{T}; \mathbb{R}^d)} \quad (3.7)$$

holds for all  $t \in [0, T]$ .

On the other hand, if  $v$  belongs to the class  $Z_T^{m,\delta}$ , we see  $v(t) \in C(\mathbb{T}; (w(N))_\delta)$  and  $\|v_x(t)\|_{H^m(\mathbb{T}; \mathbb{R}^d)} \leq 2M$  follows for all  $t \in [0, T]$ . Thus, noting the form of  $F(v)$  and the compactness of  $w(N)$ , it is easy to check that there exists  $C_2 = C_2(a, b, M, N) > 0$  such that

$$\|F(\pi \circ v)\|_{H^{m-2}(\mathbb{T}; \mathbb{R}^d)} \leq C_2 \|v_x\|_{H^m(\mathbb{T}; \mathbb{R}^d)}, \quad (3.8)$$

$$\|F(\pi \circ u) - F(\pi \circ v)\|_{H^{m-2}(\mathbb{T}; \mathbb{R}^d)} \leq C_2 \|u_x - v_x\|_{H^m(\mathbb{T}; \mathbb{R}^d)} \quad (3.9)$$

for any  $u, v \in Z_T^{m,\delta}$ .

Using the smoothing property (3.7) and the nonlinear estimates (3.8) and (3.9), we can prove  $L$  is a contraction mapping from  $Z_{T_\varepsilon}^{m,\delta}$  into itself provided that  $T_\varepsilon$  is sufficiently small. It is the standard argument, thus we omit the detail.  $\square$

*Proof of Lemma 3.3.* Suppose  $v \in Y_{T_\varepsilon}^{m,\delta}$  solves (3.5)- (3.6). Define  $\rho : (w(N))_\delta \rightarrow \mathbb{R}^d$  by  $\rho(Q) = Q - \pi(Q)$  for  $Q \in (w(N))_\delta$ . Then it follows from the definition that  $|\rho \circ v(t, x)| = \min_{Q' \in w(N)} |v(t, x) - Q'|$  since  $w(N)$  is compact. In addition,  $(v(t) - w \circ u_0)$  belongs to  $L^\infty(\mathbb{T}; \mathbb{R}^d)$  since  $v \in Y_{T_\varepsilon}^{m,\delta}$ . Thus  $\rho \circ v(t)$  makes sense in  $L^2(\mathbb{T}; \mathbb{R}^d)$  for each  $t$ . To obtain that  $v$  is  $w(N)$ -valued, we show

$$\|\rho \circ v(t)\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 = \langle \rho \circ v(t), \rho \circ v(t) \rangle = 0$$

for all  $t \in [0, T_\varepsilon]$ . Since  $\pi + \rho$  is identity on  $(w(N))_\delta$ ,

$$d\pi_v + d\rho_v = I_d \quad (3.10)$$

follows on  $T_v(w(N))_\delta$ , where  $I_d$  is the identity. By identifying  $T_v(w(N))_\delta$  with  $\mathbb{R}^d$ , we see that  $v_t(t, x) \in T_{v(t, x)}(w(N))_\delta$  and  $d\pi_v(v_t)(t, x) \in T_{\pi \circ v(t, x)} w(N)$  for each  $(t, x)$ . Thus  $\langle \rho \circ v, d\pi_v(v_t) \rangle = 0$  holds. Using this relation and (3.10), we deduce

$$\frac{1}{2} \frac{d}{dt} \|\rho \circ v\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 = \langle \rho \circ v, d\rho_v(v_t) \rangle = \langle \rho \circ v, d\rho_v(v_t) + d\pi_v(v_t) \rangle = \langle \rho \circ v, v_t \rangle.$$

Here let us notice that  $(-\varepsilon(\pi \circ v)_{xxxx} + F(\pi \circ v))(t) \in \Gamma((\pi \circ v)^{-1} T w(N))$  since  $\pi \circ v(t) \in w(N)$ , and thus this is perpendicular to  $\rho \circ v(t)$ . Noting this and substituting (3.5), we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\rho \circ v\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 &= \langle \rho \circ v, -\varepsilon v_{xxxx} + F(\pi \circ v) \rangle \\ &= \langle \rho \circ v, -\varepsilon(\rho \circ v)_{xxxx} - \varepsilon(\pi \circ v)_{xxxx} + F(\pi \circ v) \rangle \\ &= \langle \rho \circ v, -\varepsilon(\rho \circ v)_{xxxx} \rangle \\ &= -\varepsilon \|(\rho \circ v)_{xx}\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 \leq 0, \end{aligned}$$

which implies  $\|\rho \circ v(t)\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 \leq \|\rho \circ v_0\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 = 0$ . Hence  $\rho \circ v(t) = 0$  holds for all  $t$ . Thus  $v(t)$  is  $w(N)$ -valued for all  $t$ . This completes the proof.  $\square$

#### 4. GEOMETRIC AND CLASSICAL ENERGY ESTIMATE

Assume that  $N$  is compact also in this section. Let  $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$  be a sequence of solutions to (3.1)-(3.2) constructed in Section 3. We will evaluate the bundle-valued Sobolev norms of  $\{u_x^\varepsilon\}_{\varepsilon \in (0,1)}$  and obtain the uniform estimate on the norm and the existence time. Our goal of this section is the following.

**Lemma 4.1.** *Let  $u_0 \in H^{m+1}(\mathbb{T}; N)$  with an integer  $m \geq 2$ , and let  $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$  be a sequence of solutions to (3.1)-(3.2). Then there exists a constant  $T > 0$  depending only on  $a, b, N, \|u_{0x}\|_{H^2(\mathbb{T}; TN)}$  such that  $\{u_x^\varepsilon\}_{\varepsilon \in (0,1)}$  is bounded in  $L^\infty(0, T; H^m(\mathbb{T}; TN))$ .*

*Proof of Lemma 4.1.* To obtain the desired uniform bounds, we show that

$$\frac{d}{dt} \|u_x^\varepsilon(t)\|_{H^2(\mathbb{T}; TN)}^2 \leq C(a, b, N) \sum_{r=4}^8 \|u_x^\varepsilon(t)\|_{H^2(\mathbb{T}; TN)}^r, \quad (4.1)$$

$$\frac{d}{dt} \|u_x^\varepsilon(t)\|_{H^k(\mathbb{T}; TN)}^2 \leq C(a, b, N, \|u_x^\varepsilon(t)\|_{H^{k-1}(\mathbb{T}; TN)}) \|u_x^\varepsilon(t)\|_{H^k(\mathbb{T}; TN)}^2, \quad 3 \leq k \leq m, \quad (4.2)$$

hold for all  $t \in [0, T_\varepsilon]$ .

Throughout the proof of (4.1) and (4.2), we simply write  $u$ ,  $J$ ,  $g$  in place of  $u^\varepsilon$ ,  $J_u$ ,  $g_u$  respectively,  $\|\cdot\|_{H^k} = \|\cdot\|_{H^k(\mathbb{T}; TN)}$ ,  $\|\cdot\|_{L^2} = \|\cdot\|_{L^2(\mathbb{T}; TN)}$ ,  $\|\cdot\|_{L^\infty} = \|\cdot\|_{L^\infty(\mathbb{T}; TN)}$  for  $k \in \mathbb{N}$ , and sometimes omit to write time variable  $t$ .

Let  $2 \leq k \leq m$ . We consider the following quantity

$$\frac{1}{2} \frac{d}{dt} \|u_x\|_{H^k}^2 = \frac{1}{2} \sum_{l=0}^k \frac{d}{dt} \|\nabla_x^l u_x\|_{L^2}^2 = \sum_{l=0}^k \int_{\mathbb{T}} g(\nabla_t \nabla_x^l u_x, \nabla_x^l u_x) dx. \quad (4.3)$$

Note that  $\nabla_t u_x = \nabla_x u_t$  and  $\nabla_t \nabla_x u_x = \nabla_x \nabla_t u_x + R(u_t, u_x)u_x$  follows from the definition of the covariant derivative, where  $R$  denotes the curvature tensor on  $(N, J, g)$ . Using these commutative relations inductively, we have for  $l \geq 1$

$$\nabla_t \nabla_x^l u_x = \nabla_x^{l+1} u_t + \sum_{j=0}^{l-1} \nabla_x^j [R(u_t, u_x) \nabla_x^{l-(j+1)} u_x]. \quad (4.4)$$

Substituting (4.4) and (3.1) into (4.3) gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u_x\|_{H^k}^2 \\ &= \sum_{l=0}^k \int_{\mathbb{T}} g(\nabla_x^{l+1} u_t, \nabla_x^l u_x) dx + \sum_{l=1}^k \sum_{j=0}^{l-1} \int_{\mathbb{T}} g(\nabla_x^j [R(u_t, u_x) \nabla_x^{l-(j+1)} u_x], \nabla_x^l u_x) dx \\ &= -\varepsilon \sum_{l=0}^k \int_{\mathbb{T}} g(\nabla_x^{l+4} u_x, \nabla_x^l u_x) dx \\ &\quad + a \sum_{l=0}^k \int_{\mathbb{T}} g(\nabla_x^{l+3} u_x, \nabla_x^l u_x) dx \\ &\quad + \sum_{l=0}^k \int_{\mathbb{T}} g(\nabla_x^{l+1} J \nabla_x u_x, \nabla_x^l u_x) dx \\ &\quad + b \sum_{l=0}^k \int_{\mathbb{T}} g(\nabla_x^{l+1} [g(u_x, u_x) u_x], \nabla_x^l u_x) dx \\ &\quad - \varepsilon \sum_{l=1}^k \sum_{j=0}^{l-1} \int_{\mathbb{T}} g(\nabla_x^j [R(\nabla_x^3 u_x, u_x) \nabla_x^{l-(j+1)} u_x], \nabla_x^l u_x) dx \\ &\quad + a \sum_{l=1}^k \sum_{j=0}^{l-1} \int_{\mathbb{T}} g(\nabla_x^j [R(\nabla_x^2 u_x, u_x) \nabla_x^{l-(j+1)} u_x], \nabla_x^l u_x) dx \end{aligned}$$



$$\begin{aligned}
& + \sum_{l=1}^k \sum_{j=0}^{l-1} \int_{\mathbb{T}} g \left( \nabla_x^j \left[ R(J \nabla_x u_x, u_x) \nabla_x^{l-(j+1)} u_x \right], \nabla_x^l u_x \right) dx \\
& + b \sum_{l=1}^k \sum_{j=0}^{l-1} \int_{\mathbb{T}} g \left( \nabla_x^j \left[ g(u_x, u_x) R(u_x, u_x) \nabla_x^{l-(j+1)} u_x \right], \nabla_x^l u_x \right) dx.
\end{aligned} \tag{4.5}$$

Note that the last term of (4.5) equals to 0 since  $R(u_x, u_x) = 0$ . We next deduce

$$\begin{aligned}
\int_{\mathbb{T}} g \left( \nabla_x^{l+4} u_x, \nabla_x^l u_x \right) dx &= \int_{\mathbb{T}} g \left( \nabla_x^{l+2} u_x, \nabla_x^{l+2} u_x \right) dx \\
&= \|\nabla_x^{l+2} u_x\|_{L^2}^2,
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
\int_{\mathbb{T}} g \left( \nabla_x^{l+3} u_x, \nabla_x^l u_x \right) dx &= - \int_{\mathbb{T}} g \left( \nabla_x^{l+2} u_x, \nabla_x^{l+1} u_x \right) dx \\
&= -\frac{1}{2} \int_{\mathbb{T}} \left[ g \left( \nabla_x^{l+1} u_x, \nabla_x^{l+1} u_x \right) \right]_x dx \\
&= 0,
\end{aligned} \tag{4.7}$$

by integrating by parts. In addition,  $\nabla_x J = J \nabla_x$  follows from the Kähler condition. Thus, by using this relation and the antisymmetry of  $J$ , we have

$$\int_{\mathbb{T}} g \left( \nabla_x^{l+1} J \nabla_x u_x, \nabla_x^l u_x \right) dx = - \int_{\mathbb{T}} g \left( J \nabla_x^{l+1} u_x, \nabla_x^{l+1} u_x \right) dx = 0. \tag{4.8}$$

Substituting (4.6), (4.7) and (4.8) into (4.5) yields

$$\frac{1}{2} \frac{d}{dt} \|u_x\|_{H^k}^2 + \varepsilon \sum_{l=0}^k \|\nabla_x^{l+2} u_x\|_{L^2}^2 = \text{I}_k + \text{II}_k + \text{III}_k + \text{IV}_k, \tag{4.9}$$

where

$$\begin{aligned}
\text{I}_k &= b \sum_{l=0}^k \int_{\mathbb{T}} g \left( \nabla_x^{l+1} [g(u_x, u_x) u_x], \nabla_x^l u_x \right) dx, \\
\text{II}_k &= a \sum_{l=1}^k \sum_{j=0}^{l-1} \int_{\mathbb{T}} g \left( \nabla_x^j \left[ R(\nabla_x^2 u_x, u_x) \nabla_x^{l-(j+1)} u_x \right], \nabla_x^l u_x \right) dx, \\
\text{III}_k &= \sum_{l=1}^k \sum_{j=0}^{l-1} \int_{\mathbb{T}} g \left( \nabla_x^j \left[ R(J \nabla_x u_x, u_x) \nabla_x^{l-(j+1)} u_x \right], \nabla_x^l u_x \right) dx, \\
\text{IV}_k &= -\varepsilon \sum_{l=1}^k \sum_{j=0}^{l-1} \int_{\mathbb{T}} g \left( \nabla_x^j \left[ R(\nabla_x^3 u_x, u_x) \nabla_x^{l-(j+1)} u_x \right], \nabla_x^l u_x \right) dx.
\end{aligned}$$

We show the desired bounds of  $\text{I}_k$ ,  $\text{II}_k$ ,  $\text{III}_k$ , and  $\text{IV}_k$  below.

**Case 1:**  $k = 2$ .

We first consider  $\text{I}_2$ . Using Hölder's inequality and the Sobolev embedding, we deduce

$$\text{I}_2 = b \int_{\mathbb{T}} g \left( \nabla_x [g(u_x, u_x) u_x], u_x \right) dx$$

$$\begin{aligned}
& + b \int_{\mathbb{T}} g \left( \nabla_x^2 [g(u_x, u_x) u_x], \nabla_x u_x \right) dx \\
& + b \int_{\mathbb{T}} g \left( \nabla_x^3 [g(u_x, u_x) u_x], \nabla_x^2 u_x \right) dx \\
& \leq C(b) \|u_x\|_{H^2}^4 + b \int_{\mathbb{T}} g \left( \nabla_x^3 [g(u_x, u_x) u_x], \nabla_x^2 u_x \right) dx.
\end{aligned}$$

Furthermore a simple calculation gives

$$\begin{aligned}
\nabla_x^3 [g(u_x, u_x) u_x] &= 2g \left( \nabla_x^3 u_x, u_x \right) u_x + g(u_x, u_x) \nabla_x^3 u_x \\
&+ 6g \left( \nabla_x^2 u_x, \nabla_x u_x \right) u_x + 6g \left( \nabla_x^2 u_x, u_x \right) \nabla_x u_x \\
&+ 6g \left( \nabla_x u_x, u_x \right) \nabla_x^2 u_x + 6g \left( \nabla_x u_x, \nabla_x u_x \right) \nabla_x u_x,
\end{aligned}$$

hence we deduce

$$\begin{aligned}
\int_{\mathbb{T}} g \left( \nabla_x^3 [g(u_x, u_x) u_x], \nabla_x^2 u_x \right) dx &= 2 \int_{\mathbb{T}} g \left( g \left( \nabla_x^3 u_x, u_x \right) u_x, \nabla_x^2 u_x \right) dx \\
&+ \int_{\mathbb{T}} g \left( g(u_x, u_x) \nabla_x^3 u_x, \nabla_x^2 u_x \right) dx \\
&+ 12 \int_{\mathbb{T}} g \left( g \left( \nabla_x^2 u_x, u_x \right) \nabla_x u_x, \nabla_x^2 u_x \right) dx \\
&+ 6 \int_{\mathbb{T}} g \left( g \left( \nabla_x u_x, u_x \right) \nabla_x^2 u_x, \nabla_x^2 u_x \right) dx \\
&+ 6 \int_{\mathbb{T}} g \left( g \left( \nabla_x u_x, \nabla_x u_x \right) \nabla_x u_x, \nabla_x^2 u_x \right) dx. \quad (4.10)
\end{aligned}$$

We see  $\nabla_x^3 u_x$  disappears from the first and the second term of the right hand side of (4.10) by a good symmetricity of  $g$ . In fact, after integrating by parts, we have

$$2 \int_{\mathbb{T}} g \left( g \left( \nabla_x^3 u_x, u_x \right) u_x, \nabla_x^2 u_x \right) dx = -2 \int_{\mathbb{T}} g \left( g \left( \nabla_x^2 u_x, u_x \right) \nabla_x u_x, \nabla_x^2 u_x \right) dx, \quad (4.11)$$

$$\int_{\mathbb{T}} g \left( g(u_x, u_x) \nabla_x^3 u_x, \nabla_x^2 u_x \right) dx = - \int_{\mathbb{T}} g \left( g \left( \nabla_x u_x, u_x \right) \nabla_x^2 u_x, \nabla_x^2 u_x \right) dx. \quad (4.12)$$

Substituting (4.11), (4.12) into (4.10) and noting

$$\int_{\mathbb{T}} g \left( g \left( \nabla_x u_x, \nabla_x u_x \right) \nabla_x u_x, \nabla_x^2 u_x \right) dx = \frac{1}{4} \int_{\mathbb{T}} [g \left( \nabla_x u_x, \nabla_x u_x \right)^2]_x dx = 0,$$

we obtain

$$\begin{aligned}
& \int_{\mathbb{T}} g \left( \nabla_x^3 [g(u_x, u_x) u_x], \nabla_x^2 u_x \right) dx \\
&= 10 \int_{\mathbb{T}} g \left( g \left( \nabla_x^2 u_x, u_x \right) \nabla_x u_x, \nabla_x^2 u_x \right) dx + 5 \int_{\mathbb{T}} g \left( g \left( \nabla_x u_x, u_x \right) \nabla_x^2 u_x, \nabla_x^2 u_x \right) dx,
\end{aligned}$$

which is bounded by  $C\|u_x\|_{H^2}^4$ . Therefore we have

$$I_2 \leq C(b) \|u_x\|_{H^2}^4. \quad (4.13)$$

Next we consider  $\Pi_2$ . A simple computation gives

$$\Pi_2 = a \int_{\mathbb{T}} g \left( \nabla_x [R(\nabla_x^2 u_x, u_x) u_x], \nabla_x^2 u_x \right) dx$$

$$\begin{aligned}
& + a \int_{\mathbb{T}} g \left( R(\nabla_x^2 u_x, u_x) \nabla_x u_x, \nabla_x^2 u_x \right) dx \\
& + a \int_{\mathbb{T}} g \left( R(\nabla_x^2 u_x, u_x) u_x, \nabla_x u_x \right) dx.
\end{aligned} \tag{4.14}$$

Moreover it follows from the definition of the covariant derivative of  $R$

$$\begin{aligned}
\nabla_x \left[ R(\nabla_x^2 u_x, u_x) u_x \right] &= (\nabla R)(u_x) (\nabla_x^2 u_x, u_x) u_x + R(\nabla_x^3 u_x, u_x) u_x \\
&+ R(\nabla_x^2 u_x, \nabla_x u_x) u_x + R(\nabla_x^2 u_x, u_x) \nabla_x u_x.
\end{aligned} \tag{4.15}$$

By noting that

$$g(R(X, Y)Z, W) = g(R(W, Z)Y, X) \tag{4.16}$$

holds for any  $X, Y, Z, W \in \Gamma(u^{-1}TN)$ , and by substituting (4.15) into (4.14), we deduce

$$\begin{aligned}
\text{II}_2 &= a \int_{\mathbb{T}} g \left( (\nabla R)(u_x) (\nabla_x^2 u_x, u_x) u_x, \nabla_x^2 u_x \right) dx \\
&+ a \int_{\mathbb{T}} g \left( R(\nabla_x^3 u_x, u_x) u_x, \nabla_x^2 u_x \right) dx \\
&+ 3a \int_{\mathbb{T}} g \left( R(\nabla_x^2 u_x, \nabla_x u_x) u_x, \nabla_x^2 u_x \right) dx \\
&+ a \int_{\mathbb{T}} g \left( R(\nabla_x^2 u_x, u_x) u_x, \nabla_x u_x \right) dx \\
&\leq C(a, N) (\|u_x\|_{H^2}^4 + \|u_x\|_{H^2}^5) + a \int_{\mathbb{T}} g \left( R(\nabla_x^3 u_x, u_x) u_x, \nabla_x^2 u_x \right) dx.
\end{aligned} \tag{4.17}$$

Here we used the fact that  $R$  and  $\nabla R$  are bounded operators since  $N$  is compact. Furthermore,  $\nabla_x^3 u_x$  disappears from the second term of the right hand side of (4.17) because of the properties of  $R$ . In fact, we deduce from the integration by parts and (4.16)

$$\begin{aligned}
a \int_{\mathbb{T}} g \left( R(\nabla_x^3 u_x, u_x) u_x, \nabla_x^2 u_x \right) dx &= a \int_{\mathbb{T}} g \left( R(\nabla_x^2 u_x, u_x) u_x, \nabla_x^3 u_x \right) dx \\
&= \frac{a}{2} \int_{\mathbb{T}} g \left( R(\nabla_x^2 u_x, u_x) u_x, \nabla_x^3 u_x \right) dx \\
&\quad - \frac{a}{2} \int_{\mathbb{T}} g \left( R(\nabla_x^3 u_x, u_x) u_x, \nabla_x^2 u_x \right) dx \\
&\quad - \frac{a}{2} \int_{\mathbb{T}} g \left( R(\nabla_x^2 u_x, \nabla_x u_x) u_x, \nabla_x^2 u_x \right) dx \\
&\quad - \frac{a}{2} \int_{\mathbb{T}} g \left( R(\nabla_x^2 u_x, u_x) \nabla_x u_x, \nabla_x^2 u_x \right) dx \\
&\quad - \frac{a}{2} \int_{\mathbb{T}} g \left( (\nabla R)(u_x) (\nabla_x^2 u_x, u_x) u_x, \nabla_x^2 u_x \right) dx \\
&= -a \int_{\mathbb{T}} g \left( R(\nabla_x^2 u_x, \nabla_x u_x) u_x, \nabla_x^2 u_x \right) dx \\
&\quad - \frac{a}{2} \int_{\mathbb{T}} g \left( (\nabla R)(u_x) (\nabla_x^2 u_x, u_x) u_x, \nabla_x^2 u_x \right) dx,
\end{aligned} \tag{4.18}$$

which is also bounded by  $C(a, N) (\|u_x\|_{H^2}^4 + \|u_x\|_{H^2}^5)$ . Thus we get

$$\text{II}_2 \leq C(a, N) (\|u_x\|_{H^2}^4 + \|u_x\|_{H^2}^5). \tag{4.19}$$

Next we compute  $\text{III}_2$ . This can be treated as the estimation of the composite function of lower order terms. Indeed, a simple computation gives

$$\begin{aligned}
\text{III}_2 &= \int_{\mathbb{T}} g \left( \nabla_x [R(J\nabla_x u_x, u_x)u_x], \nabla_x^2 u_x \right) dx \\
&\quad + \int_{\mathbb{T}} g \left( R(J\nabla_x u_x, u_x) \nabla_x u_x, \nabla_x^2 u_x \right) dx \\
&\quad + \int_{\mathbb{T}} g \left( R(J\nabla_x u_x, u_x)u_x, \nabla_x u_x \right) dx \\
&\leq C(N) (\|u_x\|_{L^\infty}^3 \|J\nabla_x u_x\|_{L^2} \|\nabla_x^2 u_x\|_{L^2} + \|u_x\|_{L^\infty}^2 \|J\nabla_x^2 u_x\|_{L^2} \|\nabla_x^2 u_x\|_{L^2} \\
&\quad + \|u_x\|_{L^\infty} \|\nabla_x u_x\|_{L^\infty} \|J\nabla_x u_x\|_{L^2} \|\nabla_x^2 u_x\|_{L^2} \\
&\quad + \|u_x\|_{L^\infty}^2 \|J\nabla_x u_x\|_{L^2} \|\nabla_x u_x\|_{L^2}) \\
&\leq C(N) (\|u_x\|_{H^2}^4 + \|u_x\|_{H^2}^5). \tag{4.20}
\end{aligned}$$

Next we compute  $\text{IV}_2$ . Using (4.16) and the Cauchy inequality, we deduce for any  $A > 0$

$$\begin{aligned}
\text{IV}_2 &= -\varepsilon \int_{\mathbb{T}} g \left( \nabla_x [R(\nabla_x^3 u_x, u_x)u_x], \nabla_x^2 u_x \right) dx \\
&\quad - \varepsilon \int_{\mathbb{T}} g \left( R(\nabla_x^3 u_x, u_x) \nabla_x u_x, \nabla_x^2 u_x \right) dx \\
&\quad - \varepsilon \int_{\mathbb{T}} g \left( R(\nabla_x^3 u_x, u_x)u_x, \nabla_x u_x \right) dx \\
&= -\varepsilon \int_{\mathbb{T}} g \left( (\nabla R)(u_x)(\nabla_x^2 u_x, u_x)u_x, \nabla_x^3 u_x \right) dx \\
&\quad - \varepsilon \int_{\mathbb{T}} g \left( R(\nabla_x^2 u_x, u_x)u_x, \nabla_x^4 u_x \right) dx \\
&\quad - \varepsilon \int_{\mathbb{T}} g \left( R(\nabla_x^2 u_x, u_x) \nabla_x u_x, \nabla_x^3 u_x \right) dx \\
&\quad - 2\varepsilon \int_{\mathbb{T}} g \left( R(\nabla_x^2 u_x, \nabla_x u_x)u_x, \nabla_x^3 u_x \right) dx \\
&\quad - \varepsilon \int_{\mathbb{T}} g \left( R(\nabla_x u_x, u_x)u_x, \nabla_x^3 u_x \right) dx. \\
&\leq \varepsilon A \|\nabla_x^4 u_x\|_{L^2}^2 + 5\varepsilon A \|\nabla_x^3 u_x\|_{L^2}^2 \\
&\quad + \frac{\varepsilon}{4A} \left\{ \|(\nabla R)(u_x)(\nabla_x^2 u_x, u_x)u_x\|_{L^2}^2 + \|R(\nabla_x^2 u_x, u_x)u_x\|_{L^2}^2 \right. \\
&\quad \left. + \|R(\nabla_x^2 u_x, u_x) \nabla_x u_x\|_{L^2}^2 + 2\|R(\nabla_x^2 u_x, \nabla_x u_x)u_x\|_{L^2}^2 + \|R(\nabla_x u_x, u_x)u_x\|_{L^2}^2 \right\} \\
&\leq \varepsilon A \|\nabla_x^4 u_x\|_{L^2}^2 + 5\varepsilon A \|\nabla_x^3 u_x\|_{L^2}^2 + \frac{C(N)}{4A} (\|u_x\|_{H^2}^6 + \|u_x\|_{H^2}^8). \tag{4.21}
\end{aligned}$$

Combining the estimates (4.13), (4.19), (4.20) and (4.21) yields

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|u_x\|_{H^2}^2 + (1-A)\varepsilon \|\nabla_x^4 u_x\|_{L^2}^2 + (1-5A)\varepsilon \|\nabla_x^3 u_x\|_{L^2}^2 + \varepsilon \|\nabla_x^2 u_x\|_{L^2}^2 \\
&\leq C(a, b, N, A) (\|u_x\|_{H^2}^4 + \|u_x\|_{H^2}^5 + \|u_x\|_{H^2}^6 + \|u_x\|_{H^2}^8).
\end{aligned}$$

Especially take  $A > 0$  as  $A < 1/5$ , then we obtain the desired inequality (4.1).

**Case 2:**  $3 \leq k \leq m$ .

Let  $3 \leq k \leq m$ . We also compute  $I_k + II_k + III_k + IV_k$  in (4.9). We can obtain the desired inequality (4.2) in the similar argument as in the case  $k = 2$ .

We first consider  $I_k$ . A simple computation gives

$$\begin{aligned}
I_k = & 2b \sum_{l=0}^k \int_{\mathbb{T}} g(g(\nabla_x^{l+1} u_x, u_x) u_x, \nabla_x^l u_x) dx \\
& + b \sum_{l=0}^k \int_{\mathbb{T}} g(g(u_x, u_x) \nabla_x^{l+1} u_x, \nabla_x^l u_x) dx \\
& + 2b \sum_{l=0}^k (l+1) \int_{\mathbb{T}} g(g(\nabla_x^l u_x, \nabla_x u_x) u_x, \nabla_x^l u_x) dx \\
& + 2b \sum_{l=0}^k (l+1) \int_{\mathbb{T}} g(g(\nabla_x^l u_x, u_x) \nabla_x u_x, \nabla_x^l u_x) dx \\
& + 2b \sum_{l=0}^k (l+1) \int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) \nabla_x^l u_x, \nabla_x^l u_x) dx \\
& + P_k,
\end{aligned} \tag{4.22}$$

where

$$P_k = b \sum_{l=0}^k \sum_{\substack{\alpha+\beta+\gamma=l+1 \\ \alpha, \beta, \gamma \geq 0 \\ \max\{\alpha, \beta, \gamma\} \leq l-1}} \frac{(l+1)!}{\alpha! \beta! \gamma!} \int_{\mathbb{T}} g(g(\nabla_x^\alpha u_x, \nabla_x^\beta u_x) \nabla_x^\gamma u_x, \nabla_x^l u_x) dx.$$

It is easy to check  $P_k$  is bounded by  $C(b) \|u_x\|_{H^{k-1}}^2 \|u_x\|_{H^k}^2$ . On the other hand,  $I_k - P_k$  can be treated in the same way as in the case  $k = 2$  by using the estimation like (4.11) and (4.12). Indeed, by integrating by parts and by applying the good structure of  $g$ , we have

$$\begin{aligned}
2 \int_{\mathbb{T}} g(g(\nabla_x^{l+1} u_x, u_x) u_x, \nabla_x^l u_x) dx &= -2 \int_{\mathbb{T}} g(g(\nabla_x^l u_x, u_x) \nabla_x u_x, \nabla_x^l u_x) dx, \\
\int_{\mathbb{T}} g(g(u_x, u_x) \nabla_x^{l+1} u_x, \nabla_x^l u_x) dx &= - \int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) \nabla_x^l u_x, \nabla_x^l u_x) dx.
\end{aligned}$$

Therefore we deduce

$$\begin{aligned}
I_k - P_k &= b \sum_{l=0}^k (4l+2) \int_{\mathbb{T}} g(g(\nabla_x^l u_x, \nabla_x u_x) u_x, \nabla_x^l u_x) dx \\
&+ b \sum_{l=0}^k (2l+1) \int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) \nabla_x^l u_x, \nabla_x^l u_x) dx \\
&\leq C|b| \sum_{l=0}^k \|u_x\|_{L^\infty} \|\nabla_x u_x\|_{L^\infty} \|\nabla_x^l u_x\|_{L^2}^2 \\
&\leq C(b) \|u_x\|_{H^2}^2 \|u_x\|_{H^k}^2.
\end{aligned}$$

Consequently, we obtain the desired boundness

$$I_k \leq C(b, \|u_x\|_{H^{k-1}}) \|u_x\|_{H^k}^2.$$

We next estimate  $II_k$ . A simple computation yields

$$\begin{aligned} II_k &= a \sum_{l=1}^k \int_{\mathbb{T}} g(R(\nabla_x^{l+1} u_x, u_x) u_x, \nabla_x^l u_x) dx \\ &\quad + a \sum_{l=1}^k (l-1) \int_{\mathbb{T}} g((\nabla R)(u_x)(\nabla_x^l u_x, u_x) u_x, \nabla_x^l u_x) dx \\ &\quad + a \sum_{l=1}^k (2l-1) \int_{\mathbb{T}} g(R(\nabla_x^l u_x, \nabla_x u_x) u_x, \nabla_x^l u_x) dx \\ &\quad + Q_k, \end{aligned}$$

where

$$\begin{aligned} Q_k &= a \sum_{l=1}^k \sum_{j=0}^{l-1} \sum_{\substack{p+q+r+s=j \\ p,q,r,s \geq 0 \\ \max\{p,q+2,r,s+l-(j+1)\} \leq l-1}} A_{p,q,r,s}^j \\ &\quad \times \int_{\mathbb{T}} g((\nabla_x^p R)(\nabla_x^{q+2} u_x, \nabla_x^r u_x) \nabla_x^{s+l-(j+1)} u_x, \nabla_x^l u_x) dx, \\ \nabla_x^p R &= \sum_{\alpha=1}^p \sum_{\substack{\alpha + \sum_{h=1}^{\alpha} p_h = p \\ p_h \geq 0}} B_{p_1, \dots, p_{\alpha}}^{\alpha} (\nabla^{\alpha} R)(\nabla_x^{p_1} u_x, \dots, \nabla_x^{p_{\alpha}} u_x) \end{aligned}$$

for some constant  $A_{p,q,r,s}^j, B_{p_1, \dots, p_{\alpha}}^{\alpha}$  if  $p \in \mathbb{N}$ , and  $\nabla_x^0 R = R$ .

On the estimation of  $II_k - Q_k$ , the property of Riemannian curvature tensor works well similarly to the estimate (4.19). Indeed, after integrating by parts, we have

$$\begin{aligned} &\int_{\mathbb{T}} g(R(\nabla_x^{l+1} u_x, u_x) u_x, \nabla_x^l u_x) dx \\ &= -\frac{1}{2} \int_{\mathbb{T}} g((\nabla R)(u_x)(\nabla_x^l u_x, u_x) u_x, \nabla_x^l u_x) dx \\ &\quad - \int_{\mathbb{T}} g(R(\nabla_x^l u_x, u_x) u_x, \nabla_x^l u_x) dx, \end{aligned}$$

thus we deduce

$$\begin{aligned} II_k - Q_k &= a \sum_{l=1}^k (l-3/2) \int_{\mathbb{T}} g((\nabla R)(u_x)(\nabla_x^l u_x, u_x) u_x, \nabla_x^l u_x) dx \\ &\quad + a \sum_{l=1}^k (2l-2) \int_{\mathbb{T}} g(R(\nabla_x^l u_x, \nabla_x u_x) u_x, \nabla_x^l u_x) dx \\ &\leq C(a, N) (\|u_x\|_{L^\infty}^3 + \|u_x\|_{L^\infty} \|\nabla_x u_x\|_{L^\infty}) \|u_x\|_{H^k}^2. \end{aligned} \tag{4.23}$$

On the other hand, on the estimation of  $Q_k$ , if the integers  $p, q, r, s \geq 0$  satisfy  $p+q+r+s = j$  and  $\max\{p, q+2, r, s+l-(j+1)\} \leq l-1$ , we can easily check that there are at most two

elements of the set  $\{p, q+2, r, s+l-(j+1)\}$  which equals to  $l-1$ , and that the others are not greater than  $l-2$ . Thus we deduce

$$Q_k \leq C(a) \sum_{l=1}^k \left( \sum_{p=0}^{l-1} \|\nabla_x^p R\|_{L^\infty} \right) \|u_x\|_{H^{l-1}}^2 \|u_x\|_{H^l}^2.$$

Here let us notice from definition that there may appear  $u_x, \dots, \nabla_x^{p-1} u_x$  in  $\nabla_x^p R$ , but there does not appear  $\nabla_x^p u_x$  in  $\nabla_x^p R$ . Noting this, it is easy to check

$$\sum_{p=0}^{l-1} \|\nabla_x^p R\|_{L^\infty} \leq C(N) \sum_{p=0}^{l-1} \sum_{r=0}^p \|u_x\|_{H^p}^r \leq C(N) \sum_{r=0}^{l-1} \|u_x\|_{H^{l-1}}^r.$$

Therefore we have

$$Q_k \leq C(a, N) \left( \sum_{r=2}^{k+1} \|u_x\|_{H^{k-1}}^r \right) \|u_x\|_{H^k}^2. \quad (4.24)$$

Thus (4.23) and (4.24) imply the desired boundness

$$\text{II}_k \leq C(a, N, \|u_x\|_{H^{k-1}}) \|u_x\|_{H^k}^2.$$

The desired boundness of  $\text{III}_k$  and  $\text{IV}_k$  also follows from the same argument as that of  $\text{III}_2$  and  $\text{IV}_2$ . Indeed, we can easily deduce

$$\text{III}_k \leq C(N, \|u_x\|_{H^{k-1}}) \|u_x\|_{H^k}^2,$$

and there exists  $C_1 > 0$  such that for any  $A > 0$

$$\text{IV}_k \leq C_1 \varepsilon A \sum_{l=0}^k \|\nabla_x^{l+2} u_x\|_{L^2}^2 + C(N, A, \|u_x\|_{H^{k-1}}) \|u_x\|_{H^k}^2.$$

Applying these estimation of  $\text{I}_k, \text{II}_k, \text{III}_k, \text{IV}_k$  to the right hand side of (4.9) leads to

$$\frac{1}{2} \frac{d}{dt} \|u_x\|_{H^k}^2 + (1 - C_1 A) \varepsilon \sum_{l=0}^k \|\nabla_x^{l+2} u_x\|_{L^2}^2 \leq C(a, b, N, A, \|u_x\|_{H^{k-1}}) \|u_x\|_{H^k}^2. \quad (4.25)$$

Thus, by taking  $A < 1/C_1$ , we obtain the desired inequality (4.2).

By using (4.1) and (4.2), we now complete the proof of Lemma 4.1. Set  $f(t) = \|u_x^\varepsilon(t)\|_{H^2}^2 + 1$ , then we have from (4.1),

$$\frac{df}{dt} \leq C(a, b, N) f^4, \quad f(0) = \|u_{0x}\|_{H^2}^2 + 1. \quad (4.26)$$

It follows from (4.26) that there exists a positive constant  $T = T(a, b, N, \|u_{0x}\|_{H^2}) > 0$  and a positive constant  $C_2 = C_2(a, b, N, \|u_{0x}\|_{H^2}) > 0$  such that

$$\|u_x^\varepsilon(t)\|_{H^2} \leq C_2 \quad (4.27)$$

holds for all  $t \in [0, T]$ . Furthermore, since (4.2) holds for  $k = 3$ , (4.27) and the Gronwall inequality implies

$$\|u_x^\varepsilon(t)\|_{H^3}^2 \leq \|u_{0x}\|_{H^3}^2 \exp(C(a, b, N, C_2)T),$$

which implies the existence of a constant  $C_3 = C_3(a, b, N, \|u_{0x}\|_{H^3}, T) > 0$  such that

$$\|u_x^\varepsilon(t)\|_{H^3} \leq C_3$$

holds for all  $t \in [0, T]$ . It is now clear that we can show, by using (4.2) inductively for each  $3 \leq k \leq m$ , the existence of a constant  $C_m = C_m(a, b, N, \|u_{0x}\|_{H^m}, T) > 0$  such that

$$\sup_{t \in [0, T]} \|u_x^\varepsilon(t)\|_{H^m} \leq C_m.$$

It is easy to find that the solution  $u^\varepsilon$  to (3.1)-(3.2) with  $\varepsilon \in (0, 1)$  must exist on the interval  $[0, T]$ . Otherwise we extend the time interval of existence to cover  $[0, T]$ , that is, we have  $T_\varepsilon \geq T$ . Thus the lemma has been proved.  $\square$

*Remark 1.*  $\{u_x^\varepsilon\}_{\varepsilon \in (0, 1)}$  gains the regularity in the following sense. That is, by applying (4.25) with  $k = m$ , and by integrating on  $[0, T]$ , we obtain

$$2(1 - C_1 A)\varepsilon \sum_{l=0}^m \|\nabla_x^{l+2} u_x^\varepsilon\|_{L^2([0, T] \times \mathbb{T})}^2 \leq C(a, b, N, A, \|u_{0x}\|_{H^m})T + \|u_{0x}\|_{H^m}^2.$$

This implies  $\{\varepsilon^{1/2} \nabla_x^m u_x^\varepsilon\}_{\varepsilon \in (0, 1)}$  is bounded in  $L^2(0, T; H^2(\mathbb{T}; TN))$ . This property will be used in the compactness argument in the next section.

## 5. PROOF OF THEOREM 1.1

We are now in a position to complete the proof of Theorem 1.1.

*Proof of Theorem 1.1.* At first assume that  $N$  is compact.

*Proof of existence.* Suppose that  $u_0 \in H^{m+1}(\mathbb{T}; N)$  with the integer  $m \geq 2$  is given. By Proposition 3.1 there exists a sequence  $\{u^\varepsilon\}_{\varepsilon \in (0, 1)}$  solving (3.1)-(3.2) for each  $\varepsilon > 0$ . Moreover, Lemma 4.1 implies there exists  $T = T(a, b, N, \|u_{0x}\|_{H^2(\mathbb{T}; TN)}) > 0$  which is independent of  $\varepsilon \in (0, 1)$  such that  $\{u_x^\varepsilon\}_{\varepsilon \in (0, 1)}$  is bounded in  $L^\infty(0, T; H^m(\mathbb{T}; TN))$ . Thus, since  $\mathbb{T}$  is compact, we have  $\{v^\varepsilon\}_{\varepsilon \in (0, 1)}$  is bounded in  $L^\infty(0, T; H^{m+1}(\mathbb{T}; \mathbb{R}^d))$ , where  $v^\varepsilon = w \circ u^\varepsilon$ . On the other hand, as stated in Remark 1 in the previous section,  $\{\varepsilon^{1/2} \nabla_x^m u_x^\varepsilon\}_{\varepsilon \in (0, 1)}$  is bounded in  $L^2(0, T; H^2(\mathbb{T}; TN))$ . Noting this, we see  $\{u_t^\varepsilon\}_{\varepsilon \in (0, 1)}$  is bounded in  $L^2(0, T; H^{m-2}(\mathbb{T}; TN))$ , which implies  $\{v^\varepsilon\}_{\varepsilon \in (0, 1)}$  is bounded in  $C^{0, 1/2}([0, T]; H^{m-2}(\mathbb{T}; \mathbb{R}^d))$ . Consequently, by interpolating the spaces  $L^\infty(0, T; H^{m+1}(\mathbb{T}; \mathbb{R}^d))$  and  $C^{0, 1/2}([0, T]; H^{m-2}(\mathbb{T}; \mathbb{R}^d))$ , we obtain that  $\{v^\varepsilon\}_{\varepsilon \in (0, 1)}$  is bounded in the class  $C^{0, \alpha}([0, T]; H^{m+1-6\alpha}(\mathbb{T}; \mathbb{R}^d))$  for any  $0 < \alpha \leq 1/2$ . Hence we see from Rellich's theorem and the Ascoli-Arzelà theorem that there exists a subsequence  $\{v^{\varepsilon(j)}\}_{j=1}^\infty$  and

$$v \in L^\infty(0, T; H^{m+1}(\mathbb{T}; \mathbb{R}^d)) \cap C([0, T]; H^m(\mathbb{T}; \mathbb{R}^d))$$

such that

$$v^{\varepsilon(j)} \xrightarrow{w^*} v \quad \text{in } L^\infty(0, T; H^{m+1}(\mathbb{T}; \mathbb{R}^d)) \quad \text{as } j \rightarrow \infty, \quad (5.1)$$

$$v^{\varepsilon(j)} \longrightarrow v \quad \text{in } C([0, T]; H^m(\mathbb{T}; \mathbb{R}^d)) \quad \text{as } j \rightarrow \infty. \quad (5.2)$$

In particular, we see from (5.2) that  $v \in C([0, T] \times \mathbb{T}; w(N))$ . Furthermore it is easy to check that  $v$  is a solution of (2.1)-(2.2) with the initial data  $w \circ u_0$ . Thus Lemma 2.1 implies that  $u = w^{-1} \circ v \in C([0, T] \times \mathbb{T}; N)$  satisfies

$$u \in L^\infty(0, T; H^{m+1}(\mathbb{T}; N)) \cap C([0, T]; H^m(\mathbb{T}; N))$$

and solves (1.1)-(1.2) with the initial data  $u_0$ , which completes the proof of the existence.  $\square$



*Proof of uniqueness.* Let  $u, v \in L^\infty(0, T; H^{m+1}(\mathbb{T}; N)) \cap C([0, T]; H^m(\mathbb{T}; N))$  be solutions of (1.1)-(1.2) such that  $u(0, x) = v(0, x)$ . Identify  $u, v$  with  $w \circ u, w \circ v$ . Then  $u$  and  $v$  satisfy (2.1)-(2.2) with  $u(0, x) = v(0, x)$ , and  $z = u - v$  makes sense as  $\mathbb{R}^d$ -valued function. Taking the difference between two equations, we have

$$z_t - a z_{xxx} = f(u, u_x, u_{xx}) - f(v, v_x, v_{xx}),$$

where

$$\begin{aligned} f(u, u_x, u_{xx}) = & a \{ [A(u)(u_x, u_x)]_x + A(u)(u_{xx} + A(u)(u_x, u_x), u_x) \} \\ & + \tilde{J}_u(u_{xx} + A(u)(u_x, u_x)) + b |u_x|^2 u_x. \end{aligned}$$

To prove that  $z = 0$ , we show that there exists a constant  $C > 0$  depending only on  $a, b, N$ , and the quantities  $\|u_x\|_{L^\infty(0, T; H^2(\mathbb{T}; \mathbb{R}^d))}$ ,  $\|v_x\|_{L^\infty(0, T; H^2(\mathbb{T}; \mathbb{R}^d))}$  such that

$$\frac{d}{dt} \|z(t)\|_{H^1(\mathbb{T}; \mathbb{R}^d)}^2 \leq C \|z(t)\|_{H^1(\mathbb{T}; \mathbb{R}^d)}^2.$$

We write  $C$  without commenting the dependence of the constant, simply write  $\|\cdot\|_{H^1} = \|\cdot\|_{H^1(\mathbb{T}; \mathbb{R}^d)}$ ,  $\|\cdot\|_{L^2}^2 = \|\cdot\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 = \langle \cdot, \cdot \rangle$  and omit to write time variable  $t$  below.

At first, since the mean value theorem shows that

$$f(u, u_x, u_{xx}) - f(v, v_x, v_{xx}) = O(|z| + |z_x| + |z_{xx}|),$$

we can easily check

$$\frac{1}{2} \frac{d}{dt} \|z\|_{L^2}^2 = \langle z, z_t \rangle \leq C \|z\|_{H^1}^2$$

by using the integration by parts. Thus we concentrate on the estimate of

$$\frac{1}{2} \frac{d}{dt} \|z_x\|_{L^2}^2 = \langle z_x, z_{xt} \rangle = - \langle z_{xx}, z_t \rangle = - \langle z_{xx}, f_a + f_J + f_b \rangle,$$

where

$$\begin{aligned} f_a = & a \left\{ u_{xxx} + [A(u)(u_x, u_x)]_x + A(u)(u_{xx} + A(u)(u_x, u_x), u_x) \right. \\ & \left. - v_{xxx} - [A(v)(v_x, v_x)]_x - A(v)(v_{xx} + A(v)(v_x, v_x), v_x) \right\}, \\ f_J = & \tilde{J}_u(u_{xx} + A(u)(u_x, u_x)) - \tilde{J}_v(v_{xx} + A(v)(v_x, v_x)), \\ f_b = & b (|u_x|^2 u_x - |v_x|^2 v_x). \end{aligned}$$

For any  $y \in w(N)$ , let  $p(y) = d\pi_y : \mathbb{R}^d \rightarrow T_y w(N)$  be the orthogonal projection onto the tangent space of  $w(N)$  at  $y$ , and define  $n(y) = I_d - p(y)$ , where  $I_d$  is the identity on  $\mathbb{R}^d$ . Note that  $p(y)$  and  $n(y)$  behaves as symmetric matrix on  $\mathbb{R}^d$  respectively.

On the estimation of  $\langle z_{xx}, f_J \rangle$ , let us notice at first

$$\tilde{J}_v(v_{xx} + A(v)(v_x, v_x)) = \tilde{J}_v p(v) v_{xx}.$$

Since  $\tilde{J}_v p(v) : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is antisymmetric, we obtain the desired boundness. Indeed,

$$\langle z_{xx}, f_J \rangle = \left\langle z_{xx}, (\tilde{J}_u p(u) - \tilde{J}_v p(v)) u_{xx} \right\rangle + \left\langle z_{xx}, \tilde{J}_v p(v) z_{xx} \right\rangle,$$

where the second term of the right hand side vanishes and the first term of the right hand side is bounded by  $C \|z\|_{H^1}^2$  by using the integration by parts and the mean value theorem.

The desired boundness of  $\langle z_{xx}, f_b \rangle$  follows from the facts that  $\|v_x\|^2 I_d : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $(v_x, \cdot) v_x : \mathbb{R}^d \rightarrow \mathbb{R}^d$  are symmetric respectively and  $v_x$  is in  $L^\infty(0, T; H^2(\mathbb{T}; \mathbb{R}^d))$ . It is not so difficult, hence we omit the detail.

Thus it suffices to consider  $\langle z_{xx}, f_a \rangle$ . From the definition of the covariant derivative along the curve and the relations  $p(u)^2 = p(u)$ ,  $p(u) = I_d - n(u)$ , we deduce

$$\begin{aligned} & u_{xxx} + [A(u)(u_x, u_x)]_x + A(u)(u_{xx} + A(u)(u_x, u_x), u_x) \\ &= p(u) [p(u)u_{xx}]_x \\ &= p(u)u_{xxx} + p(u) [p(u)]_x u_{xx} \\ &= u_{xxx} - n(u)u_{xxx} + p(u) [p(u)]_x u_{xx}. \end{aligned}$$

Roughly speaking,  $n(u)$  gains the regularity of order 1 since  $u$  is  $w(N)$ -valued. In fact, as is shown below,  $-n(u)u_{xxx} + p(u) [p(u)]_x u_{xx}$  essentially behaves as lower order term and does not cause any bad effects on the  $H^1$ -energy estimate. We first decompose by

$$\langle z_{xx}, f_a \rangle = a(A_0 + A_1 + A_2 + A_3),$$

where

$$\begin{aligned} A_0 &= \langle z_{xx}, z_{xxx} \rangle = 0, \\ A_1 &= -\langle z_{xx}, (n(u) - n(v))u_{xxx} \rangle + \langle z_{xx}, p(v) [p(u) - p(v)]_x u_{xx} \rangle, \\ A_2 &= -\langle z_{xx}, n(v)z_{xxx} \rangle + \langle z_{xx}, p(v) [p(v)]_x z_{xx} \rangle, \\ A_3 &= \langle z_{xx}, (p(u) - p(v)) [p(u)]_x u_{xx} \rangle. \end{aligned}$$

Obviously  $A_3$  is bounded by  $C\|z\|_{H^1}^2$  by using the integration by parts and the mean value theorem. In addition, since  $p(v)$  is symmetric and  $p(v)^2 = p(v)$  on  $\mathbb{R}^d$ , we deduce

$$\begin{aligned} A_2 &= -\langle z_{xx}, n(v)z_{xxx} \rangle + \langle p(v)z_{xx}, [p(v)]_x z_{xx} \rangle \\ &= -\langle z_{xx}, n(v)z_{xxx} \rangle - \langle z_{xx}, p(v)z_{xxx} \rangle \\ &= -\langle z_{xx}, z_{xxx} \rangle \\ &= 0. \end{aligned}$$

We need to estimate  $A_1$  carefully. At first, assume that there exists real-valued functions  $G^j$  defined on a neighbourhood of  $w(N)$  in  $\mathbb{R}^d$  satisfying  $\text{grad } G^j \neq 0$  for each  $j = n+1, \dots, d$  such that

$$w(N) = \{ v \mid G^{n+1}(v) = \dots = G^d(v) = 0 \}.$$

In this case  $n \in \mathbb{N}$  is the real-dimension of  $w(N)$  as the compact submanifold of  $\mathbb{R}^d$ . Note that there exists a smooth orthonormal frame  $\{\nu^{n+1}, \dots, \nu^d\}$  for the normal bundle  $(Tw(N))^\perp$  globally on  $w(N)$ . In this setting we start the estimation of  $A_1$ . It follows from the properties of  $p(v)$ ,  $n(v)$ ,  $p(u)$  and  $n(u)$  that

$$\begin{aligned} A_1 &= -\langle z_{xx}, (n(u) - n(v))u_{xxx} \rangle - \langle z_{xx}, p(v) [n(u) - n(v)]_x u_{xx} \rangle \\ &= -\langle z_{xx}, n(v)(n(u) - n(v))u_{xxx} \rangle - \langle z_{xx}, p(v) [(n(u) - n(v))u_{xx}]_x \rangle \\ &= -\langle n(v)z_{xx}, (n(u) - n(v))u_{xxx} \rangle - \langle p(v)z_{xx}, [(n(u) - n(v))u_{xx}]_x \rangle. \end{aligned} \quad (5.3)$$

On the first term of (5.3), it is important to note

$$n(v)z_{xx} = \sum_{j=n+1}^d (z_{xx}, \nu^j(v)) \nu^j(v) = O(|z_x|) \quad (5.4)$$

holds since  $v$  is  $w(N)$ -valued. Indeed, by taking the derivative of  $(v_x, \nu^j(v)) = 0$  with respect to  $x$ , we have  $(v_{xx}, \nu^j(v)) = -(v_x, [\nu^j(v)]_x)$  and thus a simple computation implies

$$(z_{xx}, \nu^j(v)) = -(z_x, [\nu^j(v)]_x) - (u_x, [\nu^j(u) - \nu^j(v)]_x) - (u_{xx}, \nu^j(u) - \nu^j(v)), \quad (5.5)$$

which is  $O(|z_x|)$ . Hence, by noting (5.4), we have

$$-\langle n(v)z_{xx}, (n(u) - n(v))u_{xxx} \rangle \leq C\|z_x\|_{L^2}\|z\|_{L^\infty}\|u_{xxx}\|_{L^2} \leq C\|z\|_{H^1}^2.$$

On the second term of (5.3), we deduce

$$\begin{aligned} -\langle p(v)z_{xx}, [(n(u) - n(v))u_{xx}]_x \rangle &= -\langle (p(v) - p(u))z_{xx}, [(n(u) - n(v))u_{xx}]_x \rangle \\ &\quad + \langle n(u)z_{xx}, [(n(u) - n(v))u_{xx}]_x \rangle \\ &\quad - \langle z_{xx}, [(n(u) - n(v))u_{xx}]_x \rangle. \end{aligned} \quad (5.6)$$

The first term of (5.6) is obviously bounded by  $C\|z\|_{H^1}^2$ . The second term of (5.6) is also bounded by  $C\|z\|_{H^1}^2$  since  $n(u)z_{xx} = O(|z_x|)$ . We consider the third term of (5.6). We have

$$\begin{aligned} &(n(u) - n(v))u_{xx} \\ &= \sum_{j=n+1}^d (u_{xx}, \nu^j(u))\nu^j(u) - \sum_{j=n+1}^d (u_{xx}, \nu^j(v))\nu^j(v) \\ &= \sum_{j=n+1}^d (u_{xx}, \nu^j(u))(\nu^j(u) - \nu^j(v)) + \sum_{j=n+1}^d (u_{xx}, \nu^j(u) - \nu^j(v))\nu^j(v) \\ &= -\sum_{j=n+1}^d (u_x, [\nu^j(u)]_x)(\nu^j(u) - \nu^j(v)) + \sum_{j=n+1}^d (u_{xx}, \nu^j(u) - \nu^j(v))\nu^j(v). \end{aligned}$$

Thus it follows that

$$\begin{aligned} &[(n(u) - n(v))u_{xx}]_x \\ &= -\sum_{j=n+1}^d (u_x, [\nu^j(u)]_x)[\nu^j(u) - \nu^j(v)]_x + \sum_{j=n+1}^d (u_{xx}, [\nu^j(u) - \nu^j(v)]_x)\nu^j(v) \\ &\quad + \sum_{j=n+1}^d (u_{xxx}, \nu^j(u) - \nu^j(v))\nu^j(v) + O(|z|). \end{aligned}$$

Moreover, noting that  $(z_{xx}, \nu^j(v)) = O(|z_x|)$  follows from (5.5), we get

$$\begin{aligned} &\sum_{j=n+1}^d \langle z_{xx}, (u_{xx}, [\nu^j(u) - \nu^j(v)]_x)\nu^j(v) \rangle \leq C\|z\|_{H^1}^2, \\ &\sum_{j=n+1}^d \langle z_{xx}, (u_{xxx}, \nu^j(u) - \nu^j(v))\nu^j(v) \rangle \leq C\|z\|_{H^1}^2. \end{aligned}$$

Thus we have only to estimate the following quantity

$$\sum_{j=n+1}^d \langle z_{xx}, (u_x, [\nu^j(u)]_x)[\nu^j(u) - \nu^j(v)]_x \rangle. \quad (5.7)$$

Here we write

$$[\nu^j(u)]_x = D^j(u)u_x, \quad j = n+1, \dots, d,$$

where  $D^j(u) = \text{grad } \nu^j(u)$  is a  $\mathbb{R}^d \times \mathbb{R}^d$ -valued function. Using this notation, we have

$$(5.7) = \sum_{j=n+1}^d \langle z_{xx}, (u_x, [\nu^j(u)]_x) (D^j(u) - D^j(v)) u_x \rangle \\ + \sum_{j=n+1}^d \langle z_{xx}, (u_x, [\nu^j(u)]_x) D^j(v) z_x \rangle. \quad (5.8)$$

The first term of (5.8) is obviously bounded by  $C\|z\|_{H^1}^2$ . On the second term of (5.8), note first that the following relation

$$\nu^j(v) = \frac{\text{grad } G^j(v)}{|\text{grad } G^j(v)|} = \text{grad} \left( \frac{G^j(v)}{|\text{grad } G^j(v)|} \right) \quad (5.9)$$

holds at  $v \in w(N)$ . Thus it follows that

$$D^j(v) = \left( \frac{\partial^2}{\partial v^\alpha \partial v^\beta} \left( \frac{G^j(v)}{|\text{grad } G^j(v)|} \right) \right)_{1 \leq \alpha, \beta \leq d},$$

which is a symmetric matrix valued. Then we deduce

$$\sum_{j=n+1}^d \langle z_{xx}, (u_x, [\nu^j(u)]_x) D^j(v) z_x \rangle = -\frac{1}{2} \sum_{j=n+1}^d \left\langle z_x, [(u_x, [\nu^j(u)]_x) D^j(v)]_x z_x \right\rangle$$

which is bounded by  $C\|z\|_{H^1}^2$ . Consequently we obtain the desired boundness of  $A_1$ .

In the general case, there may not exists any global orthonormal frame for the normal bundle  $(Tw(N))^\perp$  on  $w(N)$ . However, we can assume without loss of generality that

$$w(N) = \bigcup_{I=1}^L \Omega_I = \bigcup_{I=1}^L \{ v \mid G_I^{n+1}(v) = \dots = G_I^d(v) = 0 \}$$

for some  $L \in \mathbb{N}$  and real-valued functions  $G_I^{n+1}, \dots, G_I^d$  defined in the neighbourhood of  $\Omega_I$  in  $\mathbb{R}^d$  with  $\text{grad } G_I^j \neq 0$  for each  $j, 1 \leq I \leq L$ . Let  $\{\lambda^I\}_{I=1}^L$  be a partition of unity associated to  $\{\Omega_I\}_{I=1}^L$ . Then on each  $\Omega_I$ , there exists a smooth orthonormal frame for the normal bundle satisfying the relation like (5.9). Furthermore, we can proceed almost the same argument as above by noting  $n(u) = n(u) \sum_{I=1}^L \lambda^I(u)$  and  $[n(u)]_x = [n(u)]_x \sum_{I=1}^L \lambda^I(u)$ . It is not difficult, thus we omit the detail.

Consequently, we obtain the desired inequality

$$\frac{d}{dt} \|z(t)\|_{H^1}^2 \leq C \|z(t)\|_{H^1}^2.$$

Thus, since  $z(0) = 0$ , Gronwall's inequality implies  $z = 0$ . This completes the proof of the uniqueness.  $\square$

*Proof of the continuity in time of  $\nabla_x^m u_x$  in  $L^2(\mathbb{T}; TN)$ .* So far in our proof, we have proved the existence of a unique solution  $u \in L^\infty(0, T; H^{m+1}(\mathbb{T}; N)) \cap C([0, T]; H^m(\mathbb{T}; N))$ . Let  $v = w \circ u$ . To obtain that  $u \in C([0, T]; H^{m+1}(\mathbb{T}; N))$ , we show  $v_x \in C([0, T]; H^m(\mathbb{T}; \mathbb{R}^d))$ . Note that it follows from the definition of the covariant derivative that

$$dw_u(\nabla_x^m u_x) = \partial_x^{m+1} v + \sum_{l=2}^{m+1} \sum_{\substack{\alpha_1 + \dots + \alpha_l = m+1 \\ \alpha_i \geq 1}} B_{(\alpha_1, \dots, \alpha_l)}(v) (\partial_x^{\alpha_1} v_x, \dots, \partial_x^{\alpha_l} v_x). \quad (5.10)$$

Here  $B_{(\alpha_1, \dots, \alpha_l)}(\cdot)$  are multi-linear vector-valued functions on  $\mathbb{R}^d$ , and it is easy to check that the second term of the right hand side of (5.10) is in  $C([0, T]; L^2(\mathbb{T}; \mathbb{R}^d))$ . Thus it suffices to show that  $dw_u(\nabla_x^m u_x)$  belongs to  $C([0, T]; L^2(\mathbb{T}; \mathbb{R}^d))$ .

First of all, we can derive from the energy estimate (4.2) and the isometricity of  $w$

$$\frac{d}{dt} \|dw_{u^\varepsilon}(\nabla_x^m u_x^\varepsilon)(t)\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 = \frac{d}{dt} \|\nabla_x^m u_x^\varepsilon(t)\|_{L^2(\mathbb{T}; TN)}^2 \leq C$$

for some  $C > 0$  which is independent of  $\varepsilon \in (0, 1)$ . Therefore it follows that

$$\|dw_{u^\varepsilon}(\nabla_x^m u_x^\varepsilon)(t)\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 \leq \|dw_u(\nabla_x^m u_x)(0)\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 + Ct.$$

Letting  $\varepsilon \downarrow 0$ , we have  $dw_u(\nabla_x^m u_x)(t) \in L^2(\mathbb{T}; \mathbb{R}^d)$  makes sense for all  $t \in [0, T]$ , and

$$\|dw_u(\nabla_x^m u_x)(t)\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 \leq \|dw_u(\nabla_x^m u_x)(0)\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 + Ct,$$

which leads to

$$\limsup_{t \rightarrow 0} \|dw_u(\nabla_x^m u_x)(t)\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 \leq \|dw_u(\nabla_x^m u_x)(0)\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2. \quad (5.11)$$

Moreover, since  $v \in L^\infty(0, T; H^{m+1}(\mathbb{T}; \mathbb{R}^d)) \cap C([0, T]; H^m(\mathbb{T}; \mathbb{R}^d))$ , we see  $dw_u(\nabla_x^m u_x)(t)$  is weakly continuous in  $L^2(\mathbb{T}; \mathbb{R}^d)$ . Hence it follows that

$$\|dw_u(\nabla_x^m u_x)(0)\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 \leq \liminf_{t \rightarrow 0} \|dw_u(\nabla_x^m u_x)(t)\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2. \quad (5.12)$$

From (5.11) and (5.12), we obtain

$$\lim_{t \rightarrow 0} \|dw_u(\nabla_x^m u_x)(t)\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2 = \|dw_u(\nabla_x^m u_x)(0)\|_{L^2(\mathbb{T}; \mathbb{R}^d)}^2. \quad (5.13)$$

Consequently, it follows from (5.13) and the weak continuity of  $dw_u(\nabla_x^m u_x)(t)$  in  $L^2(\mathbb{T}; \mathbb{R}^d)$ ,  $dw_u(\nabla_x^m u_x)(t)$  is strongly continuous in  $L^2(\mathbb{T}; \mathbb{R}^d)$  at  $t = 0$ . By the uniqueness of  $u$ , we see  $dw_u(\nabla_x^m u_x)(t)$  is strongly continuous at each  $t \in [0, T]$  in the same way.  $\square$

Finally assume that  $N$  is noncompact. In this case, retake  $N'$  as a compact subset of  $N$  in which the image of initial data is contained. Then we can proceed the same argument on  $N'$  as in the case  $N$  is compact. Thus we complete the proof of Theorem 1.1.  $\square$

## 6. GLOBAL EXISTENCE

The goal of this section is to prove Theorem 1.2. Let  $(N, J, g)$  be a compact Riemann surface with constant Gaussian curvature  $K$ , and assume that  $a \neq 0$  and  $b = aK/2$ . Theorem 1.1 tells us that, given a initial data  $u_0 \in H^{m+1}(\mathbb{T}; N)$ , there exists  $T = T(a, b, N, \|u_{0x}\|_{H^2(\mathbb{T}; TN)}) > 0$  such that the IVP (1.1)-(1.2) admits a unique time-local solution  $u \in C([0, T]; H^{m+1}(\mathbb{T}; N))$ .

In what follows we will extend the existence time of  $u$  over  $[0, \infty)$ . For this, we have the following energy conversation laws.

**Lemma 6.1.** *For  $u \in C([0, T]; H^{m+1}(\mathbb{T}; N))$  solving (1.1)-(1.2), the following quantities*

$$\begin{aligned} & \|u_x(t)\|_{L^2(\mathbb{T}; TN)}^2, \\ & E(u(t)) = \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{T}; TN)}^2 + \frac{K^2}{8} \int_{\mathbb{T}} (g(u_x(t), u_x(t)))^3 dx \\ & \quad - K \int_{\mathbb{T}} (g(u_x(t), \nabla_x u_x(t)))^2 dx \end{aligned}$$

$$- \frac{3K}{2} \int_{\mathbb{T}} g(u_x(t), u_x(t)) g(\nabla_x u_x(t), \nabla_x u_x(t)) dx$$

are preserved with respect to  $t \in [0, T)$ .

*Remark 2.* In [15] and [19], Nishiyama and Tani treated (1.1)-(1.2) in case  $N = \mathbb{S}^2$  with  $K = 1$ , and proved a time-global existence theorem by using the following conserved quantity:

$$\|u_{xxx}(t)\|^2 - \frac{7}{2} \| |u_x(t)| |u_{xx}(t)| \|^2 - 14 \|u_x(t) \cdot u_{xx}(t)\|^2 + \frac{21}{8} \| |u_x(t)|^3 \|^2,$$

where  $\| \cdot \| = \| \cdot \|_{L^2(\mathbb{T}; \mathbb{R}^3)}$ .  $E(u(t))$  generalizes the above quantity. In fact, we can check that this quantity is reformulated as

$$\begin{aligned} & \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{T}; TN)}^2 + \frac{1}{8} \int_{\mathbb{T}} (g(u_x(t), u_x(t)))^3 dx \\ & - \int_{\mathbb{T}} (g(u_x(t), \nabla_x u_x(t)))^2 dx - \frac{3}{2} \int_{\mathbb{T}} g(u_x(t), u_x(t)) g(\nabla_x u_x(t), \nabla_x u_x(t)) dx, \end{aligned}$$

which is just  $E(u(t))$  with  $K = 1$ .

*Proof of Lemma 6.1.* It is obvious that  $\|u_x(t)\|_{L^2(\mathbb{T}; TN)}^2$  is preserved by the same computation as in Section 4. Hence we omit the proof.

We consider

$$\begin{aligned} E(u(t)) = & \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{T}; TN)}^2 + A \int_{\mathbb{T}} (g(u_x(t), u_x(t)))^3 dx \\ & - B \int_{\mathbb{T}} (g(u_x(t), \nabla_x u_x(t)))^2 dx \\ & - C \int_{\mathbb{T}} g(u_x(t), u_x(t)) g(\nabla_x u_x(t), \nabla_x u_x(t)) dx, \end{aligned}$$

where  $A = K^2/8$ ,  $B = K$ ,  $C = 3K/2$ . Since  $(N, J, g)$  has a constant sectional curvature  $K$  as a  $C^\infty$ -manifold, it follows for  $X, Y$ , and  $Z \in \Gamma(u^{-1}TN)$  that

$$R(X, Y)Z = K \{g(Y, Z)X - g(X, Z)Y\}. \quad (6.1)$$

Especially since  $\nabla R = 0$  holds, the term containing  $\nabla^p R$ ,  $p \in \mathbb{N}$  does not appear. We have only to compute by using (6.1). We make use of the integration by parts repeatedly. Hence we only show the results of computations. A simple computation gives

$$\begin{aligned} \frac{d}{dt} E(u) = & 2 \int_{\mathbb{T}} g(\nabla_x^3 u_t, \nabla_x^2 u_x) dx \\ & - (2B + 2C) \int_{\mathbb{T}} g(\nabla_x u_x, \nabla_x u_x) g(\nabla_x u_x, u_t) dx \\ & - (6A + 2CK) \int_{\mathbb{T}} (g(u_x, u_x))^2 g(\nabla_x u_x, u_t) dx \\ & - (2K + 2B + 4C) \int_{\mathbb{T}} g(u_x, \nabla_x^2 u_x) g(\nabla_x u_x, u_t) dx \\ & + (2K - 8C) \int_{\mathbb{T}} g(\nabla_x u_x, u_x) g(\nabla_x^2 u_x, u_t) dx \\ & - (2K + 2C) \int_{\mathbb{T}} g(u_x, u_x) g(\nabla_x^3 u_x, u_t) dx \end{aligned}$$

$$\begin{aligned}
& + (2K - 2B) \int_{\mathbb{T}} g(u_x, \nabla_x^3 u_x) g(u_x, u_t) dx \\
& - (24A - 2CK) \int_{\mathbb{T}} g(u_x, u_x) g(u_x, \nabla_x u_x) g(u_x, u_t) dx \\
& - (6B - 4C) \int_{\mathbb{T}} g(\nabla_x^2 u_x, \nabla_x u_x) g(u_x, u_t) dx.
\end{aligned}$$

We next substitute  $u_t = a \nabla_x^2 u_x + J \nabla_x u_x + b g(u_x, u_x) u_x$  into above and compute by repeating integration by parts. Then we deduce

$$\begin{aligned}
\frac{d}{dt} E(t) = & (6K - 4C) \int_{\mathbb{T}} g(\nabla_x u_x, u_x) g(\nabla_x^2 u_x, J \nabla_x u_x) dx \\
& - (2K + 4B - 4C) \int_{\mathbb{T}} g(\nabla_x u_x, \nabla_x^2 u_x) g(u_x, J \nabla_x u_x) dx \\
& - (2K - 2B) \int_{\mathbb{T}} g(u_x, \nabla_x^2 u_x) g(u_x, J \nabla_x^2 u_x) dx \\
& - (24A - 2CK) \int_{\mathbb{T}} g(u_x, u_x) g(u_x, \nabla_x u_x) g(u_x, J \nabla_x u_x) dx \\
& + \{-(4K + 6B)a + 20b\} \int_{\mathbb{T}} g(u_x, \nabla_x^2 u_x) g(\nabla_x u_x, \nabla_x^2 u_x) dx \\
& + \{(4K - 6C)a + 10b\} \int_{\mathbb{T}} g(u_x, \nabla_x u_x) g(\nabla_x^2 u_x, \nabla_x^2 u_x) dx \\
& + \{(36A + 2CK)a - 10Cb\} \int_{\mathbb{T}} g(u_x, u_x) g(u_x, \nabla_x u_x) g(\nabla_x u_x, \nabla_x u_x) dx \\
& + \{(24A - 2CK)a + (-6B + 4C)b\} \int_{\mathbb{T}} (g(\nabla_x u_x, u_x))^3 dx.
\end{aligned}$$

Since  $A = K^2/8$ ,  $B = K$ ,  $C = 3K/2$  and  $b = aK/2$ , a simple computation shows

$$\begin{aligned}
\frac{d}{dt} E(t) = & -10(Ka - 2b) \int_{\mathbb{T}} g(u_x, \nabla_x^2 u_x) g(\nabla_x^2 u_x, \nabla_x u_x) dx \\
& - 5(Ka - 2b) \int_{\mathbb{T}} g(u_x, \nabla_x u_x) g(\nabla_x^2 u_x, \nabla_x^2 u_x) dx \\
& + \frac{15}{2} K(Ka - 2b) \int_{\mathbb{T}} g(u_x, u_x) g(u_x, \nabla_x u_x) g(\nabla_x u_x, \nabla_x u_x) dx \\
= & 0,
\end{aligned}$$

which completes the proof.  $\square$

*Proof of Theorem 1.2.* Let  $u \in C([0, T]; H^{m+1}(\mathbb{T}; N))$  be the time-local solution of (1.1)-(1.2) which exists on the maximal time interval  $[0, T)$ . If  $T = \infty$ , Theorem 1.2 holds true. Thus we only need to consider the case  $T < \infty$ . From Lemma 6.1, we know that

$$\|u_x(t)\|_{L^2(\mathbb{T}; TN)}^2 = \|u_{0x}\|_{L^2(\mathbb{T}; TN)}^2, \quad E(u(t)) = E(u_0). \quad (6.2)$$

Hence it follows that

$$\|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{T}; TN)}^2 = E(u_0) - \frac{K^2}{8} \int_{\mathbb{T}} (g(u_x(t), u_x(t)))^3 dx$$

$$\begin{aligned}
& + K \int_{\mathbb{T}} (g(u_x(t), \nabla_x u_x(t)))^2 dx \\
& + \frac{3K}{2} \int_{\mathbb{T}} g(u_x(t), u_x(t)) g(\nabla_x u_x(t), \nabla_x u_x(t)) dx \\
& \leq E(u_0) + C|K| \|u_x(t)\|_{L^\infty(\mathbb{T}; TN)}^2 \|\nabla_x u_x(t)\|_{L^2(\mathbb{T}; TN)}^2.
\end{aligned}$$

Note here the Sobolev inequality and the Gagliardo-Nirenberg inequality of the form

$$\|u_x(t)\|_{L^\infty(\mathbb{T}; TN)}^2 \leq C \|u_x(t)\|_{L^2(\mathbb{T}; TN)} (\|u_x(t)\|_{L^2(\mathbb{T}; TN)} + \|\nabla_x u_x(t)\|_{L^2(\mathbb{T}; TN)}), \quad (6.3)$$

$$\|\nabla_x u_x(t)\|_{L^2(\mathbb{T}; TN)} \leq C \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{T}; TN)}^{1/2} \|u_x(t)\|_{L^2(\mathbb{T}; TN)}^{1/2} \quad (6.4)$$

hold. See e.g., [9, Lemma 1. 3. and 1. 4.]. From (6.2), (6.3) and (6.4), we deduce

$$\begin{aligned}
& \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{T}; TN)}^2 \\
& \leq E(u_0) + C|K| \|u_{0x}\|_{L^2(\mathbb{T}; TN)} \\
& \quad \times \left( \|u_{0x}\|_{L^2(\mathbb{T}; TN)} + \|u_{0x}\|_{L^2(\mathbb{T}; TN)}^{1/2} \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{T}; TN)}^{1/2} \right) \\
& \quad \times \|u_{0x}\|_{L^2(\mathbb{T}; TN)} \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{T}; TN)} \\
& \leq C(N, \|u_{0x}\|_{H^2(\mathbb{T}; TN)}) (1 + \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{T}; TN)}^{3/2}).
\end{aligned}$$

Thus  $X = X(t) = \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{T}; TN)}$  satisfies  $X^2 \leq C(1 + X^{3/2})$ , which implies

$$\sup_{t \in [0, T]} \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{T}; TN)} \leq C(N, \|u_{0x}\|_{H^2(\mathbb{T}; TN)}) \quad (6.5)$$

for some  $C = C(N, \|u_{0x}\|_{H^2(\mathbb{T}; TN)}) > 0$ . Interpolating (6.2) and (6.5) we have

$$\sup_{t \in [0, T]} \|u_x(t)\|_{H^2(\mathbb{T}; TN)} \leq C(N, \|u_{0x}\|_{H^2(\mathbb{T}; TN)}).$$

Since we obtain the  $H^2(\mathbb{T}; TN)$ -boundness of  $u_x$ ,

$$\sup_{t \in [0, T]} \|u_x(t)\|_{H^m(\mathbb{T}; TN)} \leq C(N, \|u_{0x}\|_{H^m(\mathbb{T}; TN)})$$

follows as in the proof of Theorem 1.1. Hence, for small  $0 < \sigma < T$ , there exists  $T_0 > 0$  and a time-local solution  $u_1$  of (1.1)-(1.2) on the time interval  $[0, T_0]$  with initial data  $u_1(0, x) = u(T - \sigma, x)$ . From the uniform estimate of  $\|u_x(t)\|_{H^2(\mathbb{T}; TN)}$  on  $[0, T]$ , we see  $T_0$  does not depend on  $\sigma$ . Thus, by choosing  $\sigma$  small enough, we have  $T - \sigma + T_0 > T$ . By the uniqueness theorem, we know  $u_1(t, x) = u(T - \sigma + t, x)$  for any  $t \in [0, T_0]$ . Thus  $u$  can be extended to the time interval  $[0, T - \sigma + T_0]$ , which contradicts the maximality of  $T$ .  $\square$

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